Florentin Smarandache, Giorgio Nordo, Saeid Jafari
(editors)

SuperHyperTopologies and SuperHyperStructures

with their Applications

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Preface

This volume presents some original research on SuperHyperStructures and SuperHyperTopologies, new branches of mathematics with applications in applied sciences. These frameworks extend classical notions of sets, graphs and topologies by incorporating uncertainty, indeterminacy and multivalued membership, rooted in neutrosophic and Plithogenic logic. They provide non-rigorous methods for the analysis of complex systems in which ambiguity and dynamic interactions are fundamental.

The book is largely deals with 3 main subject-matters. The first one introduces mathematical foundations, including SuperHyperTopologies, Neutrosophic and Plithogenic extensions, fuzzy and soft set frameworks, and links with Łukasiewicz algebras. The second one develops structural perspectives through SuperHyperGraphs, weighted hyperedges, fuzzy neural networks, and neutrosophic automata. The third subject-matter illustrates interdisciplinary applications in medicine, finance, agriculture, tourism, economics, and data science.

An important feature of the volume is its interdisciplinary orientation. Mathematical theories are linked to practical problems, showing how SuperHyper concepts can support decision-making, optimize resources and model uncertainty in different fields. By combining fuzzy, neutrosophic and plithogenic logics within hyperstructural frameworks, the book highlights innovative approaches to indeterminate and multilayered phenomena.

The subject-matters and content of this volume is aimed not only at researchers but also graduate students and professionals in mathematics, computer science and related disciplines. It offers a comprehensive reference and several practical research avenues, paving the way for further interdisciplinary research and applications.

In this volume, there are 25 research studies with different approaches, results and findings. In what follows we give a brief account of them.

In **Chapter 1**, “Mathematical Foundations and Structural Analysis of SuperHyper Topology Generated from Binary Relations”, P. Gnanachandra, C. Arunthathi and Saeid Jafari define and present the properties of SuperHyper Topological spaces using the nth-power set construction which allows the systematic modelling of multi-level relationship structures. In **Chapter 2**, “IndetermSuperHyperSoft Sets, IndetermForestSoft Sets, and IndetermTreeSoft Sets”, Takaaki Fujita and Florentin Smarandache propose three new and interesting extensions, i.e., IndetermSuperHyperSoft Sets, IndetermForestSoft Sets, and IndetermTreeSoft Sets and giving, among others some proper examples, definitions and a thorough investigation of how these structures extend and unify existing soft–set models. **Chapter 3**, “Unveiling the n-SuperHyperGraph: A Comprehensive Overview of Graph Evolution”, R. D. Beulah. P. Jayalakshmi, R. Angel Joy focus on some

foundational concepts, mathematical framework, and practical implications of the n-SuperHyper Graph and SuperHyperGraphs-based domination number with its real-world applications in various domains, where the goal is to identify the minimal set of vertices that can effectively dominate or control a system. In **Chapter 4**, "Neutrosophic SuperHyperStructures: Foundations, Transformations and Applications", Poornima Ramakrishnan and Gayathri Palanisamy, present and study the new notions of Neutrosophic SuperHyperStructure homomorphism, anti-homomorphism, isomorphism, and Contour Neutrosophic SuperHyperStructure. They also provide theoretical foundations for these notions and obtain their basic properties and applications. In **Chapter 5**, "Dual Perspective on Adjacency Relation and Fuzzy Coloring in Fuzzy Super Hypergraphs", K. Lalithambigai, P. Gnanachandra, S. Jafari set forth a comparative analysis of two techniques for generating fuzzy super topologies on the vertex set of a fuzzy super hypergraph. They also provide, among others, a detailed comparison of how each technique influences the construction of fuzzy open sets, their interrelations, and the resulting topologies. In **Chapter 6**, "A Concise Review on Various Concepts of SuperHyperStructures", Takaaki Fujita introduces and studies several SuperHyperStructure-based concepts by emphasizing on their extensions and potential applications. In **Chapter 7**, "Directed Superhypertree-width, Mixed Tree-width, and Bidirected Tree-width", Takaaki Fujita focuses on the introduction and investigation of four generalizations—directed hypertree-width, directed superhypertree-width, mixed tree-width (for graphs combining directed and undirected edges), and bidirected tree-width (for graphs with locally oriented endpoints). He gives well-defined notions and provides illustrative examples, and studies the relationshipsand comparative behaviors of these parameters. In **Chapter 8**, "Plithogenic Superhypertopology and SuperhyperNeutrosophic Superhypertopology", Takaaki Fujita introduces two new and useful structures—Plithogenic Superhypertopology and SuperhyperNeutrosophic Superhypertopology—on the n-th powerset of a set H. He presents heir formal definitions and shows how they generalize the existing topological and uncertain-set frameworks, and analyze their key theoretical properties. In **Chapter 9**, "HyperFuzzy Graph Neural Networks and HyperPlithogenic Graph Neural Networks: Theoretical Foundations", Takaaki Fujita introduces and explores HyperFuzzy Graphs, HyperNeutrosophic Graphs, and HyperPlithogenic Graphs, which generalize Fuzzy Graphs, Neutrosophic Graphs,and Plithogenic Graphs, respectively. He also introduces their corresponding neural network architectures: HyperFuzzy Graph Neural Networks, HyperNeutrosophic Graph Neural Networks, and HyperPlithogenic Graph Neural Networks, and provides a detailed analysis of their theoretical properties. Moreover, he, as an extension, explores SuperHyperFuzzy Graph Neural Networks, SuperHyperNeutrosophic Graph Neural Networks, and SuperHyperPlithogenic Graph Neural Networks. In **Chapter 10**, "Topology induced by Fuzzy Coloring in Fuzzy Graphs", K. Lalithambigai, P. Gnanachandra, S. Jafari and L. Affe demonstrate the technique of inducing fuzzy topology on fuzzy graph's set of vertices induced by vertex coloring. Also, they introduce the important and useful notions of color lower fuzzy approximation and color upper fuzzy approximation of fuzzy subgraphs and define the closed and open sets of the fuzzy topology generated by fuzzy chromatic partition. They also give some real-time applications of the color lower fuzzy approximation and color upper fuzzy approximation. In **Chapter 11**, "Primals Associated with Rough Sets", Mostafa K. El-

Bably, Arafa A. Nasef, and Rodyna A. Hosny study the concept of primal approximation spaces and their decisive role in advancing rough set theory. They introduce primal structures and present their basic properties. Furthermore, they introduce primal supra topology and primal approximation operators. In **Chapter 12**, "Lukasiewicz fuzzy hyper hoop subalgebras and filters in hyper hoops", N. Abirami and M. Mary Jansirani, they apply the Łukasiewicz *ƒ*uzzy set to the structure of ḥyper ḥoops and in this regard, they introduce some new notions and studies Łukasiewicz *ƒ*uzzy *ƒ*ilter of ḥyper ḥoops. In **Chapter 13,** "Single-valued Łukasiewicz Neutrosophic Super Hyper Set and its Applications", T. Gokila and M. Mary Jansirani focus on the concept of Neutrosophic in Hyper and Super Hyper set and the discussion of real-world problem with sets of multiple neutrosophic components for defining the problem more precisely. Moreover, they utilize these concepts along with Łukasiewicz theory for the reduction and elimination of indetermined or false values obtained in the data. They also provided an example and discussed it for every concept of this study. In **Chapter 14**, "Nano Topology via Hypergroup", J. Kavitha, S. Jafari and V. Ramesh introduce a new class of nano open sets which form a nano topology. They demonstrate that from hypergroup one can form nano topology and vice versa. They also established invertible subhypergroup nano topology which is coarser than nano topology through least open sets. In **Chapter 15, "**A neutrosophic topology for neutrosophic automata", Anil Kr. Ram and Anupam K. Singh study the basic topological properties of neutrosophic automata through various neutrosophic operators and explore connections between them. In **Chapter 16**, "Applications of a Few Operations on Pythagorean Neutrosophic Super Hypersoft Matrices", G. Hemalatha, A. Francina Shalini investigate the Pythagorean Neutrosophic Super Hypersoft Matrix (PNSHSM) Theory. They present several new notations (operations) such PNSHS-sub matrix, Equal PNSHSM, Null PNSHSM, Universal PNSHSM, and Complement PNSHSM. In **Chapter 17**, "Topological Properties of Neutrosophic HyperEdge-Weighted SuperHyperGraphs and their application in Social Networks, Ajoy Kanti Das, Takaaki Fujita and Krishna Bhattacharjee investigate the topological properties of Neutrosophic HyperEdge-Weighted SuperHyperGraphs (NHEWSHGs) and their applications in social networks. In effect, they introduce an algorithm which enables the conversion of data into NHEWSHGs along with practical implementation on a research collaboration network. In **Chapter 18, "**A Theoretical Framework of Neutrosophic SuperHyperStructure in Cost Accounting: A New Perspective for Academic Researchers", Hacı Mustafa PAKSOY, Mehmet Ali YÜZBAŞIOĞLU, Yunus Emre KAHRAMAN[3] and Murat KARAHAN propose a new theoretical structure based on the neutrosophic superhyperstructure theory to solve advanced problems of overhead cost allocation in cost accounting. Among others, they demonstrate how to build a neutrosophic model, and also provide proofs of fundamental properties including small-gain implementable solutions for real-world cost allocation issues. In **Chapter 19**, "Enhancing Financial Analysis and Risk Management through Neutrosophic SuperHyperFunction in Accounting", Murat KARAHAN, Yunus Emre KAHRAMAN, Murat KARAHAN, Mehmet Ali YÜZBAŞIOĞLU and Hacı Mustafa PAKSOY introduce a breakthrough analytical structure combining the Neutrosophic SuperHyperFunction (NSHF) with existing financial analysis frameworks, aimed at diminishing ambiguities in accounting practices. Indeed, they provide a unifying theoretical foundation, detailed mathematical proofs and formulations, and case studies derived from applications on the corporate financial statements that validate practical

implementations. In **Chapter 20**, "A Critical Review of Super HyperFunction and Neutrosophic SuperHyperFunction: Applications and Limitations in Modern Accounting Practices", Mehmet Ali YÜZBAŞIOĞLU, Murat KARAHAN, Hacı Mustafa PAKSOY and Yunus Emre KAHRAMAN set forth a critical analysis of the integration of two advanced mathematical constructs, Super HyperFunction and Neutrosophic SuperHyperFunction, into contemporary accounting practice. In **Chapter 21**, "Implementing Neutrosophic SuperHyperStructure in Accounting Information Systems: A Practical Framework for Financial Decision Making", Yunus Emre KAHRAMAN, Hacı Mustafa PAKSOY, Murat KARAHAN and Mehmet Ali YÜZBAŞIOĞLU propose the integration of Neutrosophic SuperHyperStructure with Accounting Information Systems (AIS). For realizing this, they introduce a new and practical framework for implementing this advanced structure in AIS, including the exact steps and methodologies involved. They also discuss other cases in this context such as financial forecasting, effective risk management, and reliable financial reporting. In **Chapter 22**, "Dynamic Workforce Optimization in Agriculture Using Smarandache Fuzzy Super Hyper Semigroups", M. Mala, M. Mary Jansirani propose an innovative approach by utilizing Smarandache fuzzy super hyper semigroups to model workforce dynamics in agriculture and optimize tasks within farmers. This approach gives a mathematical framework for real-time work optimization and skills development while taking fatigue into account for scheduling, making this an excellent fit for modern smart farming applications. In **Chapter 23**, "Fuzzy Rhotrix-based Framework for Reliable Medical Diagnosis", B. Ananda Priya, P. Gnanachandra, S. Jafari, G. Nordo explore the application of fuzzy rhotrices in medical diagnosticswith a focus on improving decision-making processes. In **Chapter 24**, "Application of SuperHyperStructures in the Management of Smart and Sustainable Tourist Destinations", Héctor Julio Real Aquino investigates the application of SuperHyperStructures in the management of smart and sustainable tourist destinations. In **Chapter 25**, "Neutrosophic Milli Topological Spaces", Basavaraj M. Ittanagi, Sunil Jacob John, Saeid Jafari and Chetana Chandraiah introduced a new hybrid structure called neutrosophic milli topology, where fuzzy milli topology and intuitionistic milli topology can also be deduced. They also obtained a classification of neutrosophic milli topology and established some properties concerning neutrosophic milli interior and neutrosophic milli closureness.

Chapter 1

Mathematical Foundations and Structural Analysis of SuperHyper Topology Generated from Binary Relations

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ABSTRACT

The idea of topology generation from binary relations is expanded in this work to SuperHyper Topology, which is distinct from standard topology due to the incorporation of higher-order relational structures. SuperHyper Topology uses nth-power set constructions to describe multi-level interactions inside a system, in contrast to classical topology, which depends on point-based open sets. Expanding upon the current digraph-based topology framework, we present a methodical way to create SuperHyper topologies that guarantee closure under arbitrary unions and finite intersections. We present a rigorous mathematical basis for extending relational topologies by utilizing these sophisticated structures, which provides fresh perspectives on intricate organized systems. We define and study the properties of SuperHyper Topological spaces using the nth-power set construction, which enables the systematic modelling of multi-level relationship structures. This study offers new insights into intricate organized systems by extending relational topologies with a rigorous mathematical base.

Keywords: SuperHyper Topology, Binary Relations

1. INTRODUCTION

Across several areas, topology is essential to comprehending related systems. The main focus of traditional topology is on point-based open sets and their connections. However, in order to properly represent higher-order interactions, conventional topological frameworks must be extended as relational complexity rises in contemporary applications. The SuperHyper Topology, which includes multi-layered relational structures developed from power sets, is one example of such an extension. By employing nth-power set structures, SuperHyper Topology expands on classical topology and enables a more thorough depiction of hierarchical and interconnected systems. This method makes it possible to investigate intricate structures whose relationships span higher-level relational groups rather than just paired interactions. The advent of SuperHyper Topology offers a potent instrument for examining network systems, structured data, and decision-making procedures.

Studies on graph-based topologies, metric-induced topologies, and relational structures in network science have made significant additions to the previous research on topology creation utilizing directed graphs (digraphs) and binary relations. Relational topology offers important insights into the structure of organized systems, as these works have shown. However, multilayered interactions are frequently missed by current

approaches, which calls for the creation of more sophisticated frameworks. Using higher-level relational structures generated from power sets, we extend these ideas to SuperHyper Topology in this study. We establish a strong mathematical basis for the study of SuperHyper Topology by formalizing its genesis from binary relations. Topological spaces that more accurately capture the hierarchical character of relational systems can be systematically constructed using this paradigm.

Multi-layered relational structures created from power sets are included into SuperHyper Topology, an extension of classical topology that enables a more thorough depiction of hierarchical and interconnected systems. Studies on graph-based topologies, metric-induced topologies, and relational structures in network science have made significant additions to the previous research on topology creation utilizing directed graphs (digraphs) and binary relations. We apply these ideas to SuperHyper Topology in this study, which includes higher-level relational structures that are formed from power sets.

Diesto and Gervacio [1] began the study of topological structures formed from graphs by investigating finite topological graphs. Gamorez et al. [2] then expanded on this research by examining topologies induced by a graph's neighborhoods under binary operations. By examining topologies that were directly induced by graphs, Gervacio and Guerrero [3] progressed this line of inquiry. Gnanachandra et al. [4] and Lalithambigai et al. [5] have taken more contemporary techniques, proposing topology generation from relations in energy systems via digraphs and studying topologies generated by graph metrics on vertex sets, respectively.

In addition, Nianga and Canoy [6] looked at topologies produced by binary and unary processes. SuperHyperfunctions [7], new topologies, and neutrosophic topologies [8] were introduced by Smarandache, who also went beyond traditional graph-based techniques. Later, he offered thorough foundations for revolutionary topologies [9], SuperHyperstructures, and neutrosophic SuperHyperstructures [10]. Smarandache et al. [11] expanded the field of generalized topological frameworks by introducing neutrosophic SuperHypersoft sets, which enhanced these directions.

2. BACKGROUND

SuperHyper Topology is an extension of classical topology that uses the nth-Power Set construction to incorporate higher-order relational structures. SuperHyper Topology offers a more sophisticated view of structured data by methodically establishing open sets in $P_n^*(H)$. This enables the study of dynamic interactions in domains including theoretical informatics, network research, and computational topology. The power set structure provides a flexible framework for simulating real-world systems because it is recursive, guaranteeing that relational complexity is maintained at every level.

We must first review key topological ideas in order to create a clearly defined SuperHyper Topology from binary relations. In discrete situations, topological structures are defined using binary relations. By expanding digraph-based topology generating techniques, we create a methodical process for building SuperHyper Topology. A collection τ of subsets of a set X that satisfies the following conditions is called a topology on X if, $\emptyset \in \tau$ and $X \in \tau$; τ is closed under arbitrary unions; τ is closed under finite intersections.

We now provide formal definitions for SuperHyper Topology, digraph-based topology, and binary relations.
A subset of X × X is a binary relation R on a set X. We state that x is connected to y under R if $(x, y) \in R$. Topological structures can be induced by binary relations, especially when directed graphs (digraphs) are utilized for analysis.

Definition 1. [4] A digraph $G = (V, E)$ consists of a vertex set V and an edge set $E \subseteq V \times V$ .Given a distance function δ on V , we define open balls as

$$B_\delta(x, r) = \{y \in V | \delta(x, y) < r\},$$

where r is a fixed radius. The collection of all such open balls forms a basis for the topology τ_G, given by

$$\tau_G = \bigcup_{x \epsilon V, r \epsilon R^+} B_\delta(x, r)$$

Definition 2. [9] A SuperHyper Topology is a topology defined on the nth-Power Set of a given non-empty set H, which excludes the empty set, denoted as $P_n^*(H)$. It is constructed recursively as follows:

1. The first power set excluding the empty set is denoted as $P^*(H)$, where: $P^*(H) = P(H)\ \emptyset$

2. The second power set is given by: $P_n^*(H) = P^*\big(P^*(\mathrm{H})\big)$ where all empty sets are excluded.

3. In general, the nth-Power Set is recursively defined as: $P_n^*(H) = P^*\big(P_{n-1}^*(\mathrm{H})\big), \mathrm{n} \geq 2$.

Definition 3. [9] Let τ_{SHT} be a family of subsets of $P_n^*(H)$. Then τ_{SHT} is called a SuperHyper Topology on $P_n^*(H)$ if it satisfies the following axioms:

- (SHT-1): $\emptyset$, $P_n^*(H) \in \tau_{SHT}$.

- (SHT-2): The intersection of any finite number of elements in τ_{SHT} belongs to τ_{SHT} .

- (SHT-3): The union of any finite or infinite number of elements in τ_{SHT} belongs to τ_{SHT} .

Thus, $(P_n^*(H), \tau_{SHT})$ is called a SuperHyper Topological Space. This hierarchical structure enables the study of complex relational interactions beyond traditional topologies.

3. Generation of SuperHyper Topology from Binary Relations

The SuperHyper Topology extends the topology generated from binary relations by incorporating hierarchical structures through the nth-PowerSet. This section formally defines the construction of SuperHyper Topology.

Definition 4. Let X be a non-empty set. The nth-Power Set excluding the empty set, denoted as $P_n^*(X)$, is defined recursively as follows:

$$P_0^*(X) = X, P_{k+1}^*(X) = P^*\big(P_k^*(X)\big)\ for\ k \geq 0.$$

where $P^*(A) = P(A)\ \emptyset$ represents the power set of A excluding the empty set.

Definition 5. Given a binary relation $R \subseteq X \times X$, we define an extended relation R_n on $P_n^*(X)$ as:

$$(A, B) \in R_n \Leftrightarrow \exists x \in A, \exists y \in B\ such\ that\ (x, y) \in R.$$

This relation ensures that higher-order structures preserve relational properties.

Definition 6. For each set $A \in P_n^*(X)$, the open SuperHyper ball is given by:

$$B_R^n(A) = \{B \in P_n^*(X) \mid (A, B) \in R_n\}.$$

These sets serve as a basis for the topology.

Definition 7. The SuperHyper Topological Space from Binary Relations τ_{SHT} on $(P_n^*(X))$ is defined as the topology generated from the extended relation (R_n) by the collection of open SuperHyper balls, satisfying the following conditions:

1. $\emptyset, P_n^*(X) \in \tau_{SHT}$.

2. If $(A, B \in P_n^*(X))$ and $((A, B) \in R_n)$, then (B) is an open set in (τ_{SHT}).

3. The union of any collection of sets in (τ_{SHT}) belongs to (τ_{SHT}).

4. The intersection of a finite number of sets in (τ_{SHT}) belongs to (τ_{SHT}).

Thus, $((P_n^*(X), \tau_{SHT}))$is called a SuperHyper Topological Space from Binary Relations, where relational properties at each level are preserved through the recursive extension.

Example 8: Let ($X = \{a, b, c, d\}$). The power set excluding the empty set is:

$P^*(X) = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$.

The second power set is given by: $P_2^*(X) = P^*\big(P^*(X)\big)$.

Example 9: Let ($X = \{a, b, c, d\}$) with the binary relation: $R = \{(a, b), (b, c), (c, d), (a, d)\}$.

The first power set relation (R_1) is defined as:

$$(A, B) \in R_1 \quad \text{if there exist} \quad a \in A, b \in B \quad \text{such that} \quad (a, b) \in R.$$

This gives:

$$R_1 = \{(\{a\}, \{b\}), (\{b\}, \{c\}), (\{c\}, \{d\}), (\{a\}, \{d\}), (\{a, c\}, \{a, d\}), (\{a, c\}, \{b, d\}), (\{a, b\}, \{b, c\})\}.$$

The second power set relation (R_2) extends $(R_1)$to sets of sets:

$$(A, B) \in R_2 \quad \text{if there exist} \quad A' \in A,\ B' \in B \quad \text{such that} \quad (A', B') \in R_1.$$

This gives:

$$R_2 = \{(\{\{a\}\}, \{\{b\}\}), (\{\{b\}\}, \{\{c\}\}), (\{\{c\}\}, \{\{d\}\}), (\{\{a\}\}, \{\{d\}\}), (\{\{a\}, \{\{c\}\}\}, \{\{a\}, \{\{d\}\}\})\}.$$

Example 10: For $P_1^*(X)$, the open SuperHyper balls are:

$$B_R^1(\{a\}) = \{\{b\}, \{d\}\}, \quad B_R^1(\{b\}) = \{\{c\}\}, \quad B_R^1(\{c\}) = \{\{d\}\}.$$

For $P_2^*(X)$, extending to higher levels:

$$B_R^2(\{\{a\}\}) = \{\{\{b\}\}, \{\{d\}\}\}, \quad B_R^2(\{\{b\}\}) = \{\{\{c\}\}\}, \quad B_R^2(\{\{c\}\}) = \{\{\{d\}\}\}.$$

Example 11: The SuperHyper Topology τ_{SHT} on $P_1^*(X)$ is:

$$\tau_{SHT} = \{\emptyset, P_1^*(X), \{\{a\}\}, \{\{b\}\}, \{\{c\}\}, \{\{d\}\}, \{\{a\}, \{b\}\}, \{\{a\}, \{d\}\}, \{\{b\}, \{c\}\}, \{\{c\}, \{d\}\}\}.$$

Extending to $P_2^*(X)$, the topology consists of:

$\tau_{SHT} = \{\emptyset, P_2^*(X), \{\{\{a\}\}\}, \{\{\{b\}\}\}, \{\{\{c\}\}\}, \{\{\{d\}\}\}, \{\{\{a\}\}, \{\{\{b\}\}\}\}, \{\{\{a\}\}, \{\{\{d\}\}\}\}, \{\{\{b\}\}, \{\{\{c\}\}\}\}, \{\{\{c\}\}, \{\{\{d\}\}\}\}\}$.

Lemma 12. Let R be a binary relation on a non-empty set X, and let τ_{SHT} be the SuperHyper Topology induced on $P_n^*(X)$. If R is reflexive, symmetric, or transitive, then the basis elements of τ_{SHT} satisfy the following:

a) If R is reflexive, every singleton set $\{A\}$ is a basis element.
b) If R is symmetric, then for every basis element $B_R^n(A)$, its inverse $B_R^n(A^{-1})$ is also a basis element.
c) If R is transitive, then for any two basis elements $B_R^n(A)$ and $B_R^n(B)$, the relational extension $B_R^n(A) \cup B_R^n(B)$ remains a basis element in τ_{SHT}.

Proof: a) Reflexive Relation

Reflexivity implies that $(x, x) \in R$ for all $x \in X$, which extends to all levels of $P_n^*(X)$. Therefore, for any $A \in P_n^*(X)$, we have $(A, A) \in R_n,\ meaning\ B_R^n(A) = \{A\}$ is a basis element.

b) Symmetric Relation

Symmetry means $(x, y) \in R \Rightarrow (y, x) \in R$, which extends to higher levels. Hence, if A is an open set, then its inverse relation set A^{-1} satisfies $B_R^n(A^{-1}) = B_R^n(A)$, proving that symmetry is preserved in basis elements.

c) Transitive Relation

Transitivity ensures that if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. This extends to higher levels as $(A, B) \in R_n$ and $(B, C) \in R_n \Rightarrow (A, C) \in R_n$. Thus, the union of any two basis elements $B_R^n(A)\ and\ B_R^n(B)$ remains a valid basis element, proving the structural coherence of τ_{SHT}.

Theorem 13. Let R be a binary relation on a non-empty set X, and let τ_{SHT} be the SuperHyper Topology generated by R on $P_n^*(X)$. The nature of τ_{SHT} depends on the properties of R as follows:

i) If R is reflexive, then every singleton $\{A\}$ is open in τ_{SHT}.
ii) If R is symmetric, then for every open set A, its reverse relation set A^{-1} is also open.
iii) If R is transitive, then the transitive closure R^* generates a finer topology τ_{SHT}^* where path-connected sets remain open.
iv) If R is an equivalence relation, then τ_{SHT} forms a partition topology, where each equivalence class is an open and a closed set.

Proof: i) Reflexive Relation R

Reflexivity implies that $(x, x) \in R$ for all $x \in X$, which extends to all levels of $P_n^*(X)$. This ensures that every singleton $\{A\}$ satisfies$(A, A) \in R_n$, making it an open set. Therefore, every element of $P_n^*(X)$ is contained in some open set, guaranteeing a strongly generated topology.

ii) Symmetric Relation R

Symmetry means$(x, y) \in R \Rightarrow (y, x) \in R$. Extending this to $P_n^*(X)$, we have $(A, B) \in R_n \Rightarrow (B, A) \in R_n$. Consequently, if A is an open set, then its reverse relation set A^{-1} is also open, implying topological symmetry.

iii) Transitive Relation R

Transitivity means $(x, y) \in R\ and\ (y, z) \in R \Rightarrow (x, z) \in R$. This extends to higher-order sets as $(A, B) \in R_n\ and\ (B, C) \in R_n \Rightarrow (A, C) \in R_n$. The induced topology τ_{SHT}^* from the transitive closure R^* results in path-connected open sets, meaning that every connected component remains open.

iv) Equivalence Relation R

An equivalence relation is reflexive, symmetric, and transitive, meaning that it generates a partition of X. The SuperHyper Topology τ_{SHT} consists of open sets corresponding to these equivalence classes. Each equivalence class is both open and closed in τ_{SHT}, forming a discrete-like structure in higher power sets.

Thus, the nature of SuperHyper Topology is directly influenced by the properties of the underlying relation R.

Example 14: Consider the set $X = \{a, b, c, d\}$ with the equivalence relation:

$$R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (c, d), (d, c)\}.$$

This relation partitions X into equivalence classes:

$$[a] = [b] = \{a, b\}, \quad [c] = [d] = \{c, d\}.$$

The first power set P_1^* of X is:

$$P_1^*(X) = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}.$$

The basis is chosen to ensure that open sets contain full equivalence classes:

$$\mathcal{B}_1 = \{\{a, b\}, \{c, d\}\}.$$

The topology consists of all unions of basis elements:

$$\tau_{SHT} = \{\emptyset, P_1^*(X), \{\{a, b\}\}, \{\{c, d\}\}, \{\{a, b\}, \{c, d\}\}\}.$$

Corollary 14. Let $(P_n^*(X), \tau_{SHT})$ be the SuperHyper Topological space generated from a binary relation R on X. Then, for all $n \geq 1$:

i) The topology at level $n-1$ is a subtopology of the topology at level n, i.e., $\tau_{SHT}^{(n-1)} \subseteq \tau_{SHT}^{(n)}$.
ii) If R is reflexive, symmetric, or transitive, then the properties of τ_{SHT} remain valid at all levels n.
iii) If R is an equivalence relation, then the partition structure of equivalence classes is preserved at each level $P_n^*(X)$.

Proof: i) Hierarchy of Topologies: Since $P_n^*(X) = P^*\left(P_{n-1}^*(X)\right)$, every element in $P_{n-1}^*(X)$ naturally extends to an element in $P_n^*(X)$. By the definition of the SuperHyper Topology, any open set in $\tau_{SHT}^{(n-1)}$ remains valid at level n, ensuring $\tau_{SHT}^{(n-1)} \subseteq \tau_{SHT}^{(n)}$.

ii) Preservation of Reflexivity, Symmetry, and Transitivity: If R is reflexive, then for any $A \in P_n^*(X)$, we have $(A, A) \in R_n$, ensuring that reflexivity remains valid at each power set level. If R is symmetric, then for every $(A, B) \in R_n$, we have $(B, A) \in R_n$, maintaining the symmetry structure at all levels. If R is transitive, then for any $(A, B) \in R_n and (B, C) \in R_n$, we have $(A, C) \in R_n$, ensuring that transitivity is inherited.

iii) Preservation of Equivalence Classes: If R is an equivalence relation, then it induces a partition on X into equivalence classes. Since each power set level $P_n^*(X)$ is built recursively, the partition structure remains valid, ensuring that equivalence classes in $P_n^*(X)$ correspond to unions of equivalence classes in $P_{n-1}^*(X)$.

Thus, the SuperHyper Topology inherits the structure of lower levels, preserving the relational properties and topological structure across different levels of power sets.

4. Interior and Closure in SuperHyper Topology from Binary Relations

Definition 15. Let $(P_n^*(X), \tau_{SHT})$ be a SuperHyper Topological space, and let $A \subseteq P_n^*(X)$. The interior of A, denoted by $\mathrm{Int}_{SHT}(A)$, is defined as:

$$\mathrm{Int}_{SHT}(A) = \cup\{ U \in \tau_{SHT} \mid U \subseteq A\}.$$

That is, $\mathrm{Int}_{SHT}(A)$ is the largest open set contained in A.

Example 16: Consider $X = \{a, b, c, d\}$ with the binary relation $R = \{(a, b), (b, c), (c, d)\}$.

The SuperHyper Topology $\tau_{SHT}\ on\ P_1^*(X)$ consists of open sets:

$$\tau_{SHT} = \{\emptyset, P_1^*(X), \{\{a\}, \{b\}\}, \{\{b\}, \{c\}\}, \{\{a\}, \{b\}, \{c\}\}\}.$$

For $A = \{\{a\}, \{b\}, \{c\}, \{d\}\}$, the interior is:

$$\mathrm{Int}_{SHT}(A) = \cup\{\{\{a\}, \{b\}, \{c\}\}\} = \{\{a\}, \{b\}, \{c\}\}.$$

Thus, $\mathrm{Int}_{SHT}(A)$ removes $\{d\}$, which does not belong to any basis element fully contained in A.

Lemma 17. Let $B_R^n(A)$ be an open SuperHyper ball in τ_{SHT}. Then, for any $A \subseteq P_n^*(X)$:

$$\mathrm{Int}_{SHT}(A) = \bigcup\{ B_R^n(A) \mid B_R^n(A) \subseteq A\}.$$

That is, the interior consists of the union of all SuperHyper balls fully contained in A.

Proof: Every open set in τ_{SHT} is formed by SuperHyper balls. The largest open subset within A must be formed by all SuperHyper balls contained in A. Hence, $\mathrm{Int}_{SHT}(A)$ is their union.

Theorem 18. Let $(P_n^*(X), \tau_{SHT})$ be a SuperHyper Topological space, and $A, B \subseteq P_n^*(X)$. Then, the interior satisfies the following properties:

(1) $\mathrm{Int}_{SHT}(A)$ is the largest open set contained in A.
(2) If $A \subseteq B$, then $\mathrm{Int}_{SHT}(A) \subseteq \mathrm{Int}_{SHT}(B)$.
(3) $\mathrm{Int}_{SHT}(A \cap B) = \mathrm{Int}_{SHT}(A) \cap \mathrm{Int}_{SHT}(B)$.
(4) A is open if and only if $A = \mathrm{Int}_{SHT}(A)$.

Proof: (1) $\mathrm{Int}_{SHT}(A)$ is the largest open set contained in A:

By definition, the interior of A is given by: $Int_{SHT}(A) = \bigcup\{U \in \tau_{SHT} \mid U \subseteq A\}$. Since each U in the union is open, their union is also open. Moreover, any open set V contained in A is included in the union, ensuring that $\mathrm{Int}_{SHT}(A)$ is the largest such open set.

(2) If $A \subseteq B$, then $\mathrm{Int}_{SHT}(A) \subseteq \mathrm{Int}_{SHT}(B)$: Since $A \subseteq B$, every open set U contained in A is also contained in B. Thus, $\bigcup\{U \in \tau_{SHT} \mid U \subseteq A\} \subseteq \bigcup\{V \in \tau_{SHT} \mid V \subseteq B\}$. This implies $\mathrm{Int}_{SHT}(A) \subseteq \mathrm{Int}_{SHT}(B)$.

(3) $\mathrm{Int}_{SHT}(A \cap B) = \mathrm{Int}_{SHT}(A) \cap \mathrm{Int}_{SHT}(B)$: From the definition, $Int_{SHT}(A \cap B) = \bigcup\{U \in \tau_{SHT} \mid U \subseteq A \cap B\}$. Since every open set contained in $A \cap B$ is also contained in both A and B, $Int_{SHT}(A \cap B) \subseteq \mathrm{Int}_{SHT}(A) \cap \mathrm{Int}_{SHT}(B)$.

Conversely, if $x \in \mathrm{Int}_{SHT}(A) \cap \mathrm{Int}_{SHT}(B)$, then there exist open sets $U_A \subseteq A \; and \; U_B \subseteq B$ such that $x \in U_A \cap U_B$, which is also open and contained in $A \cap B$. Hence, $\mathrm{Int}_{SHT}(A) \cap \mathrm{Int}_{SHT}(B) \subseteq \mathrm{Int}_{SHT}(A \cap B)$. Thus, $\mathrm{Int}_{SHT}(A \cap B) = \mathrm{Int}_{SHT}(A) \cap \mathrm{Int}_{SHT}(B)$.

(4) A is open if and only if $A = \mathrm{Int}_{SHT}(A)$. If A is open, then by definition it contains the union of all open sets inside A, meaning that $\mathrm{Int}_{SHT}(A) = A$. Conversely, if $A = \mathrm{Int}_{SHT}(A)$, then it is a union of open sets and thus open.

This completes the proof.

Definition 19. Let $(P_n^*(X), \tau_{SHT})$ be a SuperHyper Topological space, and $A \subseteq P_n^*(X)$. The closure of A, denoted by $\mathrm{Cl}_{SHT}(A)$, is defined as: $\mathrm{Cl}_{SHT}(A) = \bigcap\{C \mid C \text{ is closed in } \tau_{SHT}, A \subseteq C\}$.

Example 20: Using the same example with $X = \{a, b, c, d\}$, we have:

$$\tau_{SHT} = \{\emptyset, P_1^*(X), \{\{a\}, \{b\}\}, \{\{b\}, \{c\}\}, \{\{a\}, \{b\}, \{c\}\}\}.$$

For $A = \{\{a\}, \{b\}\}$, the closure is:

$$\mathrm{Cl}_{SHT}(A) = A \cup \{\{c\}\} = \{\{a\}, \{b\}, \{c\}\}.$$

Lemma 21. A point $x \in P_n^*(X)$ belongs to $\mathrm{Cl}_{SHT}(A)$ if and only if every SuperHyper ball centered at x intersects A, i.e.,

$$x \in \mathrm{Cl}_{SHT}(A) \Leftrightarrow B_R^n(x) \cap A \neq \emptyset \text{ for all } B_R^n(x).$$

This means that the closure consists of all limit points of A with respect to SuperHyper neighborhoods.

Proof: Suppose $x \in \mathrm{Cl}_{SHT}(A)$. We prove $B_R^n(x) \cap A \neq \emptyset$ for all SuperHyper balls. By definition, $\mathrm{Cl}_{SHT}(A)$ is the smallest closed set containing A, i.e., x is in every closed set that contains A.

Assume that there exists a SuperHyper ball $B_R^n(x)$ such that $B_R^n(x) \cap A = \emptyset$.

Then, the complement $C = P_n^*(X) \setminus A \; contains \; B_R^n(x)$, making C, an open set. Since open sets are complements of closed sets, C is closed, contradicting the assumption that $x \in \mathrm{Cl}_{SHT}(A)$. Therefore, such a SuperHyper ball $B_R^n(x)$ cannot exist, implying that every SuperHyper ball centered at x must intersect A.

Suppose $B_R^n(x) \cap A \neq \emptyset$ for all SuperHyper balls. We prove $x \in \mathrm{Cl}_{SHT}(A)$. Assume $x \notin \mathrm{Cl}_{SHT}(A)$. Then, there exists a closed set C such that $A \subseteq C$ but $x \notin C$. Since closed sets are complements of open sets, this implies the existence of an open SuperHyper neighborhood U of x such that $U \cap C = \emptyset$. Since every SuperHyper ball $B_R^n(x)$ intersects A, it follows that $B_R^n(x) \cap C \neq \emptyset$, contradicting the assumption that U is a separating open set. Hence, x must be in every closed set containing A, which means $x \in \mathrm{Cl}_{SHT}(A)$. Thus, we have shown that $x \in \mathrm{Cl}_{SHT}(A)$ if and only if every SuperHyper ball centered at x intersects A, completing the proof.

Corollary 22. If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ is an increasing chain of sets, then: $\mathrm{Cl}_{SHT}(\bigcup_i A_i) = \bigcup_i \mathrm{Cl}_{SHT}(A_i)$.

This means that the closure commutes with increasing unions in SuperHyper Topology.

Proof: We need to show that $\mathrm{Cl}_{SHT}(\bigcup_i A_i) \supseteq \bigcup_i \mathrm{Cl}_{SHT}(A_i)$. Since each $A_i \subseteq A_{i+1}$,

we have $\mathrm{Cl}_{SHT}(A_i) \subseteq \mathrm{Cl}_{SHT}(A_{i+1})$.

Taking the union, we obtain an increasing sequence of closed sets:

$\bigcup_i \mathrm{Cl}_{SHT}(A_i)$ is a closed set containing $\bigcup_i A_i$.

Since $\mathrm{Cl}_{SHT}(\bigcup_i A_i)$ is the smallest closed set containing $\bigcup_i A_i$, it must contain every $\mathrm{Cl}_{SHT}(A_i)$.

Thus, $\mathrm{Cl}_{SHT}(\bigcup_i A_i) \supseteq \bigcup_i \mathrm{Cl}_{SHT}(A_i)$.

Also to show that $\mathrm{Cl}_{SHT}(\bigcup_i A_i) \subseteq \bigcup_i \mathrm{Cl}_{SHT}(A_i)$

Let $x \in \mathrm{Cl}_{SHT}(\bigcup_i A_i)$. By definition, x belongs to every closed set containing $\bigcup_i A_i$.

Since $x \in \mathrm{Cl}_{SHT}(\bigcup_i A_i)$, every SuperHyper ball $B_R^n(x)$ $intersects$ $\bigcup_i A_i$, which means:

$B_R^n(x) \cap A_i \neq \emptyset$ for some i.

This implies that x is adherent to some A_i, so it belongs to $\mathrm{Cl}_{SHT}(A_i)$.

Since x is in one of the closures $\mathrm{Cl}_{SHT}(A_i)$, we conclude: $x \in \bigcup_i \mathrm{Cl}_{SHT}(A_i)$.

Hence, $\mathrm{Cl}_{SHT}(\bigcup_i A_i) \subseteq \bigcup_i \mathrm{Cl}_{SHT}(A_i)$.

Since we have proven both inclusions, we conclude that:

$\mathrm{Cl}_{SHT}(\bigcup_i A_i) = \bigcup_i \mathrm{Cl}_{SHT}(A_i)$.

Theorem 23. Let $(P_n^*(X), \tau_{SHT})$ be a SuperHyper Topological space, and $A, B \subseteq P_n^*(X)$. Then, the closure operator satisfies the following properties:

i) $\mathrm{Cl}_{SHT}(A)$ is the smallest closed set containing A.
ii) If $A \subseteq B$, then $\mathrm{Cl}_{SHT}(A) \subseteq \mathrm{Cl}_{SHT}(B)$.
iii) $\mathrm{Cl}_{SHT}(A \cup B) = \mathrm{Cl}_{SHT}(A) \cup \mathrm{Cl}_{SHT}(B)$.
iv) A is closed if and only if $A = \mathrm{Cl}_{SHT}(A)$.

Proof: i) Smallest Closed Set Containing A: By definition, $\mathrm{Cl}_{SHT}(A)$ is the intersection of all closed sets containing A, so it is closed in τ_{SHT} . Suppose there exists a smaller closed set C' such that $A \subseteq C' \subsetneq \mathrm{Cl}_{\mathrm{SHT}}(\mathrm{A})$. Since $\mathrm{Cl}_{SHT}(A)$ is the smallest closed set containing A, C' must be part of the intersection defining $\mathrm{Cl}_{SHT}(A)$, contradicting $C' \subsetneq \mathrm{Cl}_{SHT}(A)$. Hence, $\mathrm{Cl}_{SHT}(A)$ is the smallest closed set containing A.

ii) Monotonicity: If $A \subseteq B$, then $\mathrm{Cl}_{SHT}(A) \subseteq \mathrm{Cl}_{SHT}(B)$: Since $A \subseteq B$, every closed set containing B must also contain A. Taking the intersection of all such closed sets, we get $Cl_{SHT}(A) \subseteq Cl_{SHT}(B)$. This shows that the closure operator preserves inclusion.

iii) Union Property: $\mathrm{Cl}_{SHT}(A \cup B) = \mathrm{Cl}_{SHT}(A) \cup \mathrm{Cl}_{SHT}(B)$: Since $A \cup B$ is contained in every closed set containing A and every closed set containing B, we have $\mathrm{Cl}_{SHT}(A \cup B) \subseteq \mathrm{Cl}_{SHT}(A) \cup \mathrm{Cl}_{SHT}(B)$. Conversely, any point in $\mathrm{Cl}_{SHT}(A) \cup \mathrm{Cl}_{SHT}(B)$ belongs to at least one of these closures, meaning it belongs to every closed set containing $A \cup B$. Therefore, $\mathrm{Cl}_{SHT}(A \cup B) = \mathrm{Cl}_{SHT}(A) \cup \mathrm{Cl}_{SHT}(B)$.

iv) A Set is Closed if and Only if it Equals Its Closure: If A is closed, then by definition, it is the smallest closed set containing itself. $\mathrm{Cl}_{SHT}(A) = A$. Conversely, if $A = \mathrm{Cl}_{SHT}(A)$, then A is closed by definition.

This completes the proof.

Theorem 24. Let $(P_n^*(X), \tau_{SHT})$ be a SuperHyper Topological space generated from a binary relation R. Then, τ_{SHT} is closed under finite intersections and arbitrary unions, ensuring its structural consistency.

Proof: We prove that τ_{SHT} satisfies closure properties by induction on n, the order of the power set. The topology τ_{SHT} at $P_1^*(X)$ consists of open SuperHyper balls $B_R^1(A)$.

By definition, these are unions of subsets from $P_1^*(X)$, and their finite intersections contain elements still related under R_1 . Therefore, τ_{SHT} satisfies the required closure properties at $n = 1$.

Assume for $n = k$, the topology $\tau_{SHT}^{(k)}$ is closed under finite intersections and arbitrary unions. Consider $A, B \in P_{k+1}^*(X)$ such that A, B are open in $\tau_{SHT}^{(k+1)}$. Since $P_{k+1}^*(X) = P^*\left(P_k^*(X)\right)$, each element of A and B corresponds to subsets from $P_k^*(X)$.

By the inductive hypothesis, the finite intersection $A \cap B$ remains open in $P_k^*(X)$, ensuring $A \cap B \in \tau_{SHT}^{(k+1)}$.

Similarly, arbitrary unions of open sets in $P_k^*(X)$ remain valid in $P_{k+1}^*(X)$. By the principle of mathematical induction, the closure properties hold for all $n \geq 1$.

Hence, τ_{SHT} forms a valid topology in every power set level.

Corollary 25. If A is not open in τ_{SHT}, then it satisfies: $\mathrm{Int}_{SHT}(A) \subsetneq A \subsetneq \mathrm{Cl}_{SHT}(A)$.

Proof: By definition, the interior of A, denoted by $\mathrm{Int}_{SHT}(A)$, is the largest open set contained in A:

$$\mathrm{Int}_{SHT}(A) = \cup\{U \mid U \text{ is open in } \tau_{SHT}, U \subseteq A\}.$$

Since A is not open, there exists an open set U such that $U \subsetneq A$, implying $\mathrm{Int}_{SHT}(A) \subsetneq A$.

Similarly, the closure of A, denoted by $\mathrm{Cl}_{SHT}(A)$, is the smallest closed set containing A:

$$\mathrm{Cl}_{SHT}(A) = \cap\{C \mid C \text{ is closed in } \tau_{SHT}, A \subseteq C\}.$$

Since A is not closed, there exists a closed set C such that $A \subsetneq C$, implying $A \subsetneq \mathrm{Cl}_{SHT}(A)$.

Combining both results, we obtain: $\mathrm{Int}_{SHT}(A) \subsetneq A \subsetneq \mathrm{Cl}_{SHT}(A)$.

5. Conclusion

This work introduces a systematic method for building and evaluating SuperHyper Topological spaces, extending the idea of topology generation from binary relations to SuperHyper Topology. This approach allows for a richer comprehension of multi-level interactions within structured datasets and generalizes conventional topology by integrating higher-order relational structures. In order to show how SuperHyper Topology improves on classical topology while offering a more expressive framework for relational data analysis, important features like interior and closure were studied together with their characterizations, lemmas, and corollaries. The paper demonstrates how SuperHyper Topology improves the representation of multi-layered and hierarchical systems, making it a practical tool for network science, computational topology, and structured decision-making applications.

Future studies can look into extensions to fuzzy and neutrosophic contexts as well as computer methods for effectively determining SuperHyper Topological qualities. The usefulness of SuperHyper Topology in practical issues can be further demonstrated via network science applications, especially in the modelling of hierarchical structures. Furthermore, research on how SuperHyper Topology functions in dynamical systems, optimization issues, and decision-making frameworks may pave the way for new developments in theory and practice. A more comprehensive understanding of the theoretical significance of SuperHyper Topology might be possible if this study were expanded to encompass ideas like connectedness, separation axioms, and compactness. This study establishes a solid basis for further research into generalized topological structures and transdisciplinary applications.

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Chapter 2

IndetermSuperHyperSoft Sets, IndetermForestSoft Sets, and IndetermTreeSoft Sets

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Abstract

Uncertainty modeling has been explored through diverse frameworks, including Fuzzy Sets [1], Neutrosophic Sets [2,3], Rough Sets [4,5], and Plithogenic Sets [6,7]. Soft Sets [8,9] provide a parameterized perspective, giving rise to numerous extensions such as MultiSoft Sets [10], Double Framed Soft Sets [11, 12], Hypersoft Sets [13], SuperHyperSoft Sets [14], TreeSoft Sets [15], ForestSoft Sets [16], IndetermSoft Sets, and IndetermHyperSoft Sets [17].

Research on Soft Sets plays a significant role in decision science and related areas; however, investigations into IndetermSoft Sets remain at an early stage. In this paper, we propose three novel extensions—*IndetermSuperHyperSoft Sets*, *IndetermForestSoft Sets*, and *IndetermTreeSoft Sets*—providing rigorous definitions, illustrative examples, and a comprehensive study of how these structures extend and unify existing soft–set models. In particular, an *IndetermSoft Set* captures uncertainty by mapping attribute values to subsets of a universal set where indeterminacy arises either in the attributes themselves or in their corresponding images.

Keywords: Soft Set, Hypersoft set, Superhypersoft set, IndetermSoft Set, Treesoft set

1 Introduction

1.1 Soft Set Theory

Since classical set theory is not well suited to handling uncertainty, extended frameworks such as the Fuzzy Set [1] and the Soft Set [9] have been developed. A *Soft Set* (F, E) associates each parameter in a set E with a subset of a universal set U (equivalently, a mapping $F\colon E \to \mathcal{P}(U)$). This provides a flexible framework for approximating objects in U [8,9,18]. Soft Sets offer a versatile mathematical tool for modeling uncertainty, vagueness, and parameterization, serving as a simple yet effective basis for decision making, data analysis, and knowledge representation.

Owing to this versatility, numerous extensions of Soft Sets are being investigated. For instance, an *IndetermSoft Set* models uncertain relationships between parameters and subsets, permitting indeterminacy in parameters, subsets, or their mappings simultaneously [19–22]. Other well-known related notions include the *HyperSoft Set* [23, 24], the *TreeSoft Set* [25, 26], and the *SuperHyperSoft Set* [14,27,28]. These frameworks are highly useful for managing the complex uncertainties that arise in real-world systems.

1.2 Motivation and Our Contribution

Research on Soft Sets plays a significant role in decision science and related areas; however, investigations into IndetermSoft Sets remain at an early stage. In this paper, we propose three novel extensions—*IndetermSuperHyperSoft Sets*, *IndetermForestSoft Sets*, and *IndetermTreeSoft Sets*—and provide rigorous definitions, illustrative examples, and a comprehensive study of how these structures extend and unify existing soft-set models. In particular, an *IndetermSoft Set* captures uncertainty by mapping attribute values to subsets of a universal set while allowing indeterminacy to arise either in the attributes themselves, in their corresponding images, or in both.

2 Preliminaries

This section introduces the foundational concepts and definitions necessary for the discussions presented in this paper. Throughout this paper, all sets under consideration are assumed to be undirected, finite, and simple—i.e., they do not contain multiple elements representing the same entity.

2.1 Soft Set, Hypersoft set, and Superhypersoft set

A Soft Set provides a simplified approach to parameterized decision modeling by mapping attributes, or parameters, to subsets of a universal set, thereby addressing uncertainty in a straightforward manner [8,9,29]. Expanding on this idea, a Hypersoft Set facilitates enhanced multi-attribute decision analysis by mapping combinations of multiple attributes to subsets of a universal set [23, 24, 30]. Further generalizing the principles of Hypersoft Sets, SuperHypersoft Sets extend their functionality by mapping power set combinations of multiple attribute values to subsets of a universal set. This approach enables higher-dimensional decision-making and accommodates complex interrelationships among attributes [31–34].

Definition 2.1 (Soft Set)**.** [8, 9] Let U be a universal set and A be a set of attributes. A soft set over U is a pair $(\mathcal{F}, S)$, where $S \subseteq A$ and $\mathcal{F} : S \to \mathcal{P}(U)$. Here, $\mathcal{P}(U)$ denotes the power set of U. Mathematically, a soft set is represented as:

$$(\mathcal{F}, S) = \{(\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S, \mathcal{F}(\alpha) \in \mathcal{P}(U)\}.$$

Each $\alpha \in S$ is called a parameter, and $\mathcal{F}(\alpha)$ is the set of elements in U associated with α.

Definition 2.2 (Hypersoft Set)**.** [24] Let U be a universal set, and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be attribute domains. Define $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m$, the Cartesian product of these domains. A hypersoft set over U is a pair $(G, \mathcal{C})$, where $G : \mathcal{C} \to \mathcal{P}(U)$. The hypersoft set is expressed as:

$$(G, \mathcal{C}) = \{(\gamma, G(\gamma)) \mid \gamma \in \mathcal{C}, G(\gamma) \in \mathcal{P}(U)\}.$$

For an m-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathcal{C}$, where $\gamma_i \in \mathcal{A}_i$ for $i = 1, 2, \ldots, m$, $G(\gamma)$ represents the subset of U corresponding to the combination of attribute values $\gamma_1, \gamma_2, \ldots, \gamma_m$.

Example 2.3 (Real-life example of a Hypersoft Set)**.** Let the universal set of cars be

$$U = \{\text{Car1}, \text{Car2}, \text{Car3}, \text{Car4}\}.$$

Suppose there are two attribute domains:

- $\mathcal{A}_1 = \{\text{Petrol}, \text{Electric}\}$ (fuel type),
- $\mathcal{A}_2 = \{\text{Sedan}, \text{SUV}\}$ (car body type).

Then the Cartesian product of attributes is

$$\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 = \{(\text{Petrol}, \text{Sedan}), (\text{Petrol}, \text{SUV}), (\text{Electric}, \text{Sedan}), (\text{Electric}, \text{SUV})\}.$$

Define the mapping $G : \mathcal{C} \to \mathcal{P}(U)$ as follows:

$$\begin{aligned} G(\text{Petrol}, \text{Sedan}) &= \{\text{Car1}\}, \\ G(\text{Petrol}, \text{SUV}) &= \{\text{Car2}\}, \\ G(\text{Electric}, \text{Sedan}) &= \{\text{Car3}\}, \\ G(\text{Electric}, \text{SUV}) &= \{\text{Car4}\}. \end{aligned}$$

Thus, the hypersoft set $(G, \mathcal{C})$ models the decision process by assigning subsets of cars to combinations of multiple attributes (fuel type and body type). This allows more precise classification than ordinary soft sets, which handle only single attributes at a time.

Definition 2.4 (SuperHyperSoft Set)**.** [14] Let U be a universal set, and let $\mathcal{P}(U)$ denote the power set of U. Consider n distinct attributes $a_1, a_2, \ldots, a_n$, where $n \geq 1$. Each attribute a_i is associated with a set of attribute values A_i, satisfying the property $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Define $\mathcal{P}(A_i)$ as the power set of A_i for each $i = 1, 2, \ldots, n$. Then, the Cartesian product of the power sets of attribute values is given by:

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n).$$

A SuperHyperSoft Set over U is a pair $(F, \mathcal{C})$, where:

$$F : \mathcal{C} \to \mathcal{P}(U),$$

and F maps each element $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{C}$ (with $\alpha_i \in \mathcal{P}(A_i)$) to a subset $F(\alpha_1, \alpha_2, \ldots, \alpha_n) \subseteq U$. Mathematically, the SuperHyperSoft Set is represented as:

$$(F, \mathcal{C}) = \{(\gamma, F(\gamma)) \mid \gamma \in \mathcal{C}, F(\gamma) \in \mathcal{P}(U)\}.$$

Here, $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{C}$, where $\alpha_i \in \mathcal{P}(A_i)$ for $i = 1, 2, \ldots, n$, and $F(\gamma)$ corresponds to the subset of U defined by the combined attribute values $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Example 2.5 (Real-life example of a SuperHyperSoft Set)**.** Consider a university admission system where the universal set is

$$U = \{\text{Yamada}, \text{Suzuki}, \text{Tanaka}, \text{Kobayashi}, \text{Sato}\},$$

the set of applicants. Suppose there are three attributes:

- $A_1 = \{\text{Science, Engineering}\}$ (field of study),
- $A_2 = \{\text{Undergraduate, Graduate}\}$ (program level),
- $A_3 = \{\text{Scholarship, Self-funded}\}$ (financial support).

For each attribute A_i, we take the power set $\mathcal{P}(A_i)$. An element of the Cartesian product

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3)$$

could be, for instance, $(\{\text{Science, Engineering}\}, \{\text{Graduate}\}, \{\text{Scholarship}\})$.

Define a mapping

$$F : \mathcal{C} \to \mathcal{P}(U),$$

such that $F(\{\text{Science, Engineering}\}, \{\text{Graduate}\}, \{\text{Scholarship}\}) = \{\text{Yamada}, \text{Tanaka}\}$, meaning that Yamada and Tanaka are graduate applicants in both Science and Engineering fields who applied for scholarships.

Thus, the SuperHyperSoft Set $(F, \mathcal{C})$ models complex admission criteria by allowing combinations of subsets of attribute values rather than single values, enabling flexible and higher-dimensional decision-making in the university admission process.

2.2 Treesoft set

A *TreeSoft Set* maps the power set of a hierarchical tree-like structure of attributes, Tree(A), to subsets of a universal set.

$$F : P(\text{Tree}(A)) \to P(H),$$

where Tree(A) consists of multi-level attributes forming a tree-like graph [15, 17, 35, 36]. It is known that concepts such as TreeSoft Sets can generalize MultiSoft Sets and related frameworks.

Definition 2.6. [19] Let U be a universe of discourse, and let H be a non-empty subset of U, with $P(H)$ denoting the power set of H. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes (parameters, factors, etc.), for some integer $n \geq 1$, where each attribute A_i (for $1 \leq i \leq n$) is considered a first-level attribute.

Each first-level attribute A_i consists of sub-attributes, defined as:

$$A_i = \{A_{i,1}, A_{i,2}, \ldots\},$$

where the elements $A_{i,j}$ (for $j = 1, 2, \ldots$) are second-level sub-attributes of A_i. Each second-level sub-attribute $A_{i,j}$ may further contain sub-sub-attributes, defined as:

$$A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, \ldots\},$$

and so on, allowing for as many levels of refinement as needed. Thus, we can define sub-attributes of an m-th level with indices $A_{i_1,i_2,\ldots,i_m}$, where each i_k (for $k = 1, \ldots, m$) denotes the position at each level.

This hierarchical structure forms a tree-like graph, which we denote as Tree(A), with root A (level 0) and successive levels from 1 up to m, where m is the depth of the tree. The terminal nodes (nodes without descendants) are called *leaves* of the graph-tree.

A *TreeSoft Set* F is defined as a function:

$$F : P(\text{Tree}(A)) \to P(H),$$

where Tree(A) represents the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and $P(\text{Tree}(A))$ denotes its power set.

Example 2.7 (Real-life example of a TreeSoft Set)**.** Consider a company recruitment system where the universe is

$$U = \{\text{Yamada}, \text{Suzuki}, \text{Tanaka}, \text{Kobayashi}, \text{Sato}\},$$

the set of applicants. Let the attributes form a hierarchical tree structure:

- A_1 = Education
  - $A_{1,1}$ = Undergraduate
  - $A_{1,2}$ = Graduate
    - $A_{1,2,1}$ = Master
    - $A_{1,2,2}$ = Doctorate
- A_2 = Department
  - $A_{2,1}$ = Engineering
  - $A_{2,2}$ = Business

This hierarchical attribute structure $\mathrm{Tree}(A)$ allows us to express decisions using different levels of refinement.

For example, define

$$F(\{A_{1,2,1}, A_{2,1}\}) = \{\text{Yamada, Tanaka}\},$$

which means that Yamada and Tanaka are applicants with a Master's degree applying to the Engineering department.

Thus, the TreeSoft Set $(F, \mathrm{Tree}(A))$ models recruitment decisions by incorporating hierarchical attributes, enabling multi-level and structured evaluation of applicants.

2.3 ForestSoft Set

A *ForestSoft Set* is formed by taking a collection of TreeSoft Sets and "gluing" (uniting) them together so as to obtain a single function whose domain is the union of all tree-nodes' power sets and whose values in $P(H)$ combine the images given by the individual TreeSoft Sets [16, 37, 38].

Definition 2.8 (ForestSoft Set)**.** [16] Let U be a universe of discourse, $H \subseteq U$ be a non-empty subset, and $P(H)$ be the power set of H. Suppose we have a finite (or countable) collection of TreeSoft Sets

$$\big\{ F_t : P(\mathrm{Tree}(A^{(t)})) \to P(H) \big\}_{t \in T},$$

where each F_t is a TreeSoft Set corresponding to a tree $\mathrm{Tree}(A^{(t)})$ of attributes $A^{(t)}$.

We construct a *forest* by taking the (disjoint) union of all these trees:

$$\mathrm{Forest}\big(\{A^{(t)}\}_{t \in T}\big) \;=\; \bigsqcup_{t \in T} \mathrm{Tree}\big(A^{(t)}\big).$$

A *ForestSoft Set*, denoted by

$$\mathbf{F} : P\big(\mathrm{Forest}(\{A^{(t)}\})\big) \;\longrightarrow\; P(H),$$

is defined as the *union* of all TreeSoft Set mappings F_t. Concretely, for any element $X \in P\big(\mathrm{Forest}(\{A^{(t)}\})\big)$, we set

$$\mathbf{F}(X) \;=\; \bigcup_{\substack{t \in T \\ X \cap \mathrm{Tree}(A^{(t)}) \neq \varnothing}} F_t\big(X \cap \mathrm{Tree}(A^{(t)})\big),$$

where we only apply F_t to that portion of X belonging to the tree $\mathrm{Tree}(A^{(t)})$.

Example 2.9 (Cross-Role Candidate Assessment with a ForestSoft Set)**.** Let

$$U = \{\text{Alice, Bob, Charlie, Diana}\}, \qquad H = U.$$

We have two roles $t = 1, 2$, with attribute trees:

$$\mathrm{Tree}(A^{(1)}) : \begin{cases} \text{Technical Skills} & = \{\text{Programming, Algorithms}\}, \\ \text{Experience} & = \{\text{Junior, Senior}\}, \end{cases}$$

$$\mathrm{Tree}(A^{(2)}) : \begin{cases} \text{Creativity} & = \{\text{Sketching, Visualization}\}, \\ \text{Portfolio} & = \{\text{Online, Physical}\}. \end{cases}$$

Thus $\mathrm{Forest} = \mathrm{Tree}(A^{(1)}) \sqcup \mathrm{Tree}(A^{(2)})$ and $P(\mathrm{Forest})$ is its power set.

Define two TreeSoft Sets

$$F_1 : P\big(\mathrm{Tree}(A^{(1)})\big) \to P(H), \quad F_2 : P\big(\mathrm{Tree}(A^{(2)})\big) \to P(H)$$

by their nonempty values:

$$\begin{aligned} F_1(\{\text{Programming}\}) &= \{\text{Alice, Bob}\},\\ F_1(\{\text{Algorithms}\}) &= \{\text{Bob, Charlie}\},\\ F_1(\{\text{Programming, Senior}\}) &= \{\text{Alice}\},\\ \\ F_2(\{\text{Sketching}\}) &= \{\text{Charlie, Diana}\},\\ F_2(\{\text{Visualization}\}) &= \{\text{Alice}\},\\ F_2(\{\text{Visualization, Online}\}) &= \{\text{Diana}\}. \end{aligned}$$

The *ForestSoft Set* $\mathbf{F} : P(\text{Forest}) \to P(H)$ is defined by

$$\mathbf{F}(X) = \bigcup_{\substack{t=1,2 \\ X\cap \text{Tree}(A^{(t)})\neq\varnothing}} F_t\big(X \cap \text{Tree}(A^{(t)})\big).$$

Concrete computation. Take

$$X = \{\text{Programming, Visualization}\}.$$

Then

$$X \cap \text{Tree}(A^{(1)}) = \{\text{Programming}\}, \quad X \cap \text{Tree}(A^{(2)}) = \{\text{Visualization}\},$$

so

$$\mathbf{F}(X) = F_1(\{\text{Programming}\}) \ \cup \ F_2(\{\text{Visualization}\}) = \{\text{Alice, Bob}\} \ \cup \ \{\text{Alice}\} = \{\text{Alice, Bob}\}.$$

Thus $\mathbf{F}(X)$ yields the candidates Alice and Bob when selecting both a programming skill and a creative visualization skill across roles.

2.4 IndetermSoft set and IndetermHyperSoft Set

An *IndetermSoft Set* models uncertainty by associating attribute values with subsets of a universal set where indeterminacy exists in attributes or images. And an *IndetermHyperSoft Set* generalizes IndetermSoft Sets to multiple distinct attributes, mapping combinations of indeterminate attribute values to subsets of a universal set.

$$F : A_1 \times A_2 \times \cdots \times A_n \to P(H),$$

where indeterminacy can occur in any A_i, $P(H)$, or $F(a_1, a_2, \ldots, a_n)$ [19–22].

Definition 2.10 (IndetermSoft set)**.** Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the powerset of H. Let A be the set of attribute values for an attribute a. A function $F : A \to P(H)$ is called an *IndetermSoft Set* if at least one of the following conditions holds:

1. A has some indeterminacy.
2. $P(H)$ has some indeterminacy.
3. There exists at least one $v \in A$ such that $F(v)$ is indeterminate (unclear, uncertain, or not unique).
4. Any two or all three of the above conditions.

An IndetermSoft Set is represented mathematically as:

$$F : A \to H(\cap, \cup, \oplus, \neg),$$

where $H(\cap, \cup, \oplus, \neg)$ represents a structure closed under the IndetermSoft operators.

Example 2.11 (IndetermSoft Set: Product Quality Assessment)**.** Let

$$U = \{\text{ProdA, ProdB, ProdC}\}, \qquad H = U,$$

and let the single attribute *Quality* have values

$$A = \{\text{High, Medium, Low}\}.$$

Define

$$F : A \longrightarrow P(H)$$

by

$$F(\text{High}) = \{\text{ProdA, ProdB}\},$$
$$F(\text{Medium}) = \{\text{ProdB, } \widetilde{\text{ProdC}}\},$$
$$F(\text{Low}) = \{\widetilde{\text{ProdA}}\}.$$

Here $\widetilde{\text{ProdC}}$ and $\widetilde{\text{ProdA}}$ denote that the inclusion of ProdC (in "Medium") and ProdA (in "Low") is *indeterminate* (e.g. conflicting user reviews).

Verification of IndetermSoft conditions:

(i) A itself is determinate, but

(ii) the codomain $P(H)$ contains ambiguous membership symbols $\widetilde{\cdot}$, and

(iii) for $v = \text{Medium}$, the set $F(v)$ is not uniquely determined.

Thus (F, A) is an IndetermSoft Set modeling uncertain product ratings.

Definition 2.12 (IndetermHyperSoft Set)**.** Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the powerset of H. Let $a_1, a_2, \ldots, a_n$ $(n \geq 1)$ be n distinct attributes, with attribute values $A_1, A_2, \ldots, A_n$, such that $A_i \cap A_j = \emptyset$ for $i \neq j$. The pair $(F, A_1 \times A_2 \times \cdots \times A_n)$, where

$$F : A_1 \times A_2 \times \cdots \times A_n \to P(H),$$

is called an *IndetermHyperSoft Set* if:

1. Any A_i or $P(H)$ exhibits indeterminacy.
2. For $(a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \cdots \times A_n$, $F(a_1, a_2, \ldots, a_n)$ is indeterminate.

Example 2.13 (IndetermHyperSoft Set: Customer Purchase Patterns)**.** Let

$$U = \{\text{Alice, Bob, Charlie}\}, \qquad H = U.$$

Consider two attributes:

$$A_1 = \{\text{Online, InStore}\}, \quad A_2 = \{\text{Credit, Cash}\},$$

with $\mathcal{C} = A_1 \times A_2$. Define

$$F : \mathcal{C} \longrightarrow P(H)$$

by

$$F(\text{Online, Credit}) = \{\text{Alice, } \widetilde{\text{Bob}}\},$$
$$F(\text{Online, Cash}) = \{\text{Bob}\},$$
$$F(\text{InStore, Credit}) = \{\widetilde{\text{Charlie}}\},$$
$$F(\text{InStore, Cash}) = \{\text{Alice, Charlie}\}.$$

Here $\widetilde{\text{Bob}}$ and $\widetilde{\text{Charlie}}$ indicate that Bob's purchase channel "Online+Credit" and Charlie's "InStore+Credit" are *indeterminate* (e.g. missing transaction logs).

Verification of IndetermHyperSoft conditions:

(i) Some codomain values $F(\gamma)$ contain $\widetilde{\cdot}$, showing *ambiguous* subsets.

(ii) For $\gamma = (\text{Online, Credit})$, $F(\gamma)$ is not uniquely determined.

Hence $(F, A_1 \times A_2)$ is an IndetermHyperSoft Set capturing uncertain customer behavior.

2.5 Incomplete Soft Set

The incomplete soft set is known as a special case of the IndetermSoft set, and the following definition of the incomplete soft set has been introduced [39].

Definition 2.14. [39] An *incomplete soft set* is defined as a tuple:

$$S = (U, E, f),$$

where:

- U is a non-empty set of objects, referred to as the universe.
- E is a non-empty set of parameters.
- $f : E \to \mathcal{P}(U) \cup \{*\}$, a mapping from the parameter set E to the power set of U ($\mathcal{P}(U)$) or the special symbol $*$.

Here, $*$ indicates missing or undefined information for a given parameter $e \in E$. If $f(e) = *$, the information associated with the parameter e is incomplete. Otherwise, $f(e) \subseteq U$ as in a standard soft set.

Example 2.15 (Customer Feedback with Incomplete Data)**.** Let

$$U = \{\text{Alice, Bob, Charlie}\}$$

be the set of customers of an online retailer, and let

$$E = \{\text{PriceConcern, QualityConcern, SupportConcern}\}$$

be three feedback parameters. Define

$$f : E \longrightarrow \mathcal{P}(U) \cup \{*\}$$

by

$$\begin{aligned} f(\text{PriceConcern}) &= \{\text{Alice, Charlie}\}, \\ f(\text{QualityConcern}) &= *, \\ f(\text{SupportConcern}) &= \{\text{Bob}\}. \end{aligned}$$

Here

$$S = \big(U,\, E,\, f\big)$$

is an *incomplete soft set* because for the parameter QualityConcern no feedback was collected (hence $f(\text{QualityConcern}) = *$), while the other parameters map to the subsets of customers who raised those concerns.

Theorem 2.16. *An incomplete soft set is a specific instance of an IndetermSoft set, where indeterminacy exists only in the mapping*

$$f : E \to \mathcal{P}(U) \cup \{*\}$$

, and this indeterminacy is explicitly represented by the symbol $*$, denoting missing or undefined data.

Proof. Let $S = (U, E, f)$ be an incomplete soft set, where:

- U is a non-empty universe of discourse,
- E is a non-empty set of parameters,
- $f : E \to \mathcal{P}(U) \cup \{*\}$ is a mapping such that for each $e \in E$, $f(e) \in \mathcal{P}(U)$ or $f(e) = *$, where $*$ indicates missing or undefined data.

Let $F : A \to P(H)$ define an IndetermSoft set, where:

- A is the set of attributes,
- $P(H)$ is the powerset of a non-empty subset $H \subseteq U$,
- Indeterminacy occurs in A, $P(H)$, or $F(a)$ for $a \in A$.

To prove that S is a special case of F, we demonstrate the following correspondences:

For the incomplete soft set S, the mapping $f : E \to \mathcal{P}(U) \cup \{*\}$ satisfies:

$$f(e) = \begin{cases} A_e \subseteq U, & \text{if } f(e) \text{ is defined,} \\ *, & \text{if } f(e) \text{ is undefined.} \end{cases}$$

This corresponds to the mapping $F(a)$ of an IndetermSoft set, where:

$$F(a) \in \begin{cases} P(H), & \text{if } F(a) \text{ is determinate,} \\ \text{indeterminate}, & \text{if } F(a) \text{ is undefined or unclear.} \end{cases}$$

Here, the indeterminacy in $F(a)$ for some $a \in A$ is equivalent to $f(e) = *$ in S, thus establishing that S inherits the indeterminacy of F in the restricted case of missing data.

The IndetermSoft set allows indeterminacy in A, $P(H)$, or $F(a)$. For S, the indeterminacy is confined to $f(e)$, specifically when $f(e) = *$. This restriction implies that:

$$\text{Indeterminacy in } S \implies \text{Indeterminacy in } F,$$

but the converse does not hold, as S does not allow indeterminacy in E or $\mathcal{P}(U)$.

The incomplete soft set S is a subset of IndetermSoft sets where:

$$\forall e \in E, \quad f(e) \in \mathcal{P}(U) \cup \{*\}, \quad * \text{ explicitly represents missing data.}$$

This explicit representation of missing data is a strict subset of the broader forms of indeterminacy allowed in an IndetermSoft set, making S a specialized structure.

Since the incomplete soft set S satisfies the mapping and indeterminacy conditions of an IndetermSoft set under the restricted case of missing data, S is a special case of F. □

3 Results of This Paper

In this section, we present the main results of this paper, including the definitions and concise mathematical analysis of the IndetermTreeSoft Set, IndetermForestSoft Set, and IndetermSuperHyperSoft Set.

3.1 IndetermTreeSoft Set

The IndetermTreeSoft Set integrates the concepts of TreeSoft Sets and IndetermSoft Sets into a unified framework. Its definition and related details are provided below.

Definition 3.1 (IndetermTreeSoft Set)**.** Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the power set of H. Let $\text{Tree}(A)$ be the hierarchical tree of attributes (as defined in the TreeSoft Set, with levels of sub-attributes) and $P(\text{Tree}(A))$ its power set.

A mapping

$$F \ : \ P(\text{Tree}(A)) \ \longrightarrow \ P(H)$$

is called an *IndetermTreeSoft Set* if one or more of the following conditions of indeterminacy hold:

1. The tree of attributes, $\mathrm{Tree}(A)$, includes some indeterminate or uncertain nodes (e.g., an attribute node has unspecified or overlapping sub-attributes).
2. The codomain $P(H)$ contains uncertain or partially unknown subsets (e.g., membership of some elements of H is not uniquely determined).
3. For at least one $X \in P(\mathrm{Tree}(A))$, the image $F(X) \subseteq H$ itself is indeterminate or ambiguous.

Mathematically, one may allow each value $F(X)$ to be an element of a more general structure $H(\cap, \cup, \oplus, \neg)$ under partial or fuzzy membership (if desired), provided it accommodates indeterminacy.

Example 3.2 (Smartphone Selection with Indeterminacy)**.** Let

$$U = \{\text{PhoneA, PhoneB, PhoneC}\}$$

be a universe of smartphone models, and set $H = U$. Define a two-level attribute tree

$$\mathrm{Tree}(A): \quad \begin{matrix} \text{Features} \\ \begin{cases} \text{Performance} = \{\text{CPU}, \text{RAM}\}, \\ \text{Camera} = \{\text{DayMode, NightMode}\}. \end{cases} \end{matrix}$$

Thus $\mathrm{Tree}(A)$ has nodes $\{\text{CPU}, \text{RAM}, \text{DayMode}, \text{NightMode}\}$, and

$$P\big(\mathrm{Tree}(A)\big)$$

is its power set.

Define $F : P(\mathrm{Tree}(A)) \to P(H)$ by

$$\begin{aligned} F\big(\{\text{CPU}\}\big) &= \{\text{PhoneA, PhoneB}\}, \\ F\big(\{\text{NightMode}\}\big) &= \{\text{PhoneA, } \tilde{\text{Phone}}\text{C}\}, \\ F\big(\{\text{CPU, NightMode}\}\big) &= \{\text{PhoneA}\}. \end{aligned}$$

Here $\tilde{\text{Phone}}$C denotes that PhoneC's NightMode performance is *indeterminate* (e.g. conflicting test results), so it is unclear whether PhoneC should be included.

Verification of indeterminacy conditions:

(i) *Tree-structure indeterminacy.* The attribute node NightMode is itself *uncertain*, since different reviewers report incompatible ratings.

(ii) *Codomain indeterminacy.* The codomain value $F(\{\text{NightMode}\})$ includes the symbol $\tilde{\text{Phone}}$C, indicating an *ambiguous* membership in $P(H)$.

(iii) *Image indeterminacy.* For $X = \{\text{NightMode}\}$, the set $F(X)$ is not uniquely defined because of PhoneC's unclear status.

Thus F satisfies Definition 3.1 and is a concrete IndetermTreeSoft Set modeling smartphone selection under real-world uncertainty.

Theorem 3.3 (IndetermTreeSoft Set as a Generalization of IndetermSoft Set and TreeSoft Set)**.** *Let U be a universe and $H \subseteq U$ non-empty. Let* $\mathrm{Tree}(A)$ *be the attribute tree and* $P(\mathrm{Tree}(A))$ *its power set. Then:*

1. *Every IndetermSoft Set $F_0 : A \to P(H)$ can be embedded as an IndetermTreeSoft Set on the one-level tree* $\mathrm{Tree}(A) = A$.

2. *Every TreeSoft Set* $F : P(\mathrm{Tree}(A)) \to P(H)$ *with no indeterminacy is a (fully determinate) IndetermTreeSoft Set.*

Proof. **(1) From IndetermSoft to IndetermTreeSoft.** Let $F_0 : A \to P(H)$ be an IndetermSoft Set. Define a trivial attribute tree by taking

$$\mathrm{Tree}(A) = A \quad \Longrightarrow \quad P\big(\mathrm{Tree}(A)\big) = P(A).$$

Extend F_0 to

$$F : P(A) \longrightarrow P(H), \qquad F(X) = \bigcup_{\alpha \in X} F_0(\alpha),$$

for every $X \subseteq A$. We check the IndetermTreeSoft indeterminacy conditions:

(i) *Tree-structure indeterminacy.* If the original A contains any uncertain or overlapping attribute values, then $\mathrm{Tree}(A)$ inherits that indeterminacy.

(ii) *Codomain indeterminacy.* If $P(H)$ admits fuzzy or ambiguous subsets, then for any X, $F(X) \subseteq H$ may be indeterminate.

(iii) *Image indeterminacy.* Since $F(\{\alpha\}) = F_0(\alpha)$, any α with $F_0(\alpha)$ indeterminate yields $F(\{\alpha\})$ indeterminate.

Because at least (iii) holds whenever F_0 is indeterminate, F is an IndetermTreeSoft Set.

(2) From TreeSoft to IndetermTreeSoft. Let $F : P(\mathrm{Tree}(A)) \to P(H)$ be a deterministic TreeSoft Set, so that no uncertainty arises in the domain tree, the codomain, or any image $F(X)$. The definition of IndetermTreeSoft Set requires *at least one* indeterminacy condition, but we allow the degenerate case of "zero" indeterminacy to cover fully determinate frameworks. Hence F is a valid IndetermTreeSoft Set viewed as a special case in which none of (i)–(iii) occurs.

Thus the class of IndetermTreeSoft Sets simultaneously generalizes both IndetermSoft Sets (via item 1) and TreeSoft Sets (via item 2). □

Theorem 3.4 (Closure under Union)**.** *Let*

$$F_1,\, F_2 : P\big(\mathrm{Tree}(A)\big) \longrightarrow P(H)$$

be two IndetermTreeSoft Sets on the same attribute tree $\mathrm{Tree}(A)$ *and universe* H*. Define*

$$(F_1 \vee F_2)(X) = F_1(X) \cup F_2(X), \qquad \forall\, X \in P\big(\mathrm{Tree}(A)\big).$$

Then $F_1 \vee F_2$ *is again an IndetermTreeSoft Set.*

Proof. We must check that $F = F_1 \vee F_2$ inherits at least one of the three indeterminacy conditions from Definition 3.1.

(i) Tree-structure indeterminacy. If $\mathrm{Tree}(A)$ contains uncertain or undefined nodes, this property is independent of F, hence holds for F.

(ii) Codomain indeterminacy. If $P(H)$ admits ambiguous or fuzzy subsets, then since $\cup$ of any such subsets remains within the same fuzzy/indeterminate class, $F(X) \in P(H)$ can be indeterminate whenever $F_1(X)$ or $F_2(X)$ is.

(iii) Image indeterminacy. Suppose there exists $X_0 \subseteq \mathrm{Tree}(A)$ such that at least one of $F_1(X_0)$ or $F_2(X_0)$ is indeterminate. Then

$$F(X_0) = F_1(X_0) \cup F_2(X_0)$$

is also indeterminate (the union of an indeterminate set with any other remains indeterminate).

Because at least one of (i),(ii),(iii) holds, F satisfies the definition of an IndetermTreeSoft Set. □

Theorem 3.5 (Closure under Intersection)**.** *With the same notation as above, define*

$$(F_1 \wedge F_2)(X) \;=\; F_1(X) \;\cap\; F_2(X).$$

Then $F_1 \wedge F_2$ is an IndetermTreeSoft Set.

Proof. Again let $F = F_1 \wedge F_2$. The tree-structure (i) and codomain (ii) indeterminacies are inherited as before. For (iii), if there is some X_0 with $F_1(X_0)$ or $F_2(X_0)$ indeterminate, then their intersection

$$F(X_0) = F_1(X_0) \cap F_2(X_0)$$

remains indeterminate (intersecting an indeterminate set with any set yields indeterminacy in the resulting membership). Hence F meets the indeterminacy requirement. □

Theorem 3.6 (Restriction to Subtrees)**.** *Let F be an IndetermTreeSoft Set on* Tree(A)*, and let $T' \subseteq$ Tree(A) be any connected subtree. Define the* restriction

$$F|_{T'} : P(T') \;\longrightarrow\; P(H), \qquad F|_{T'}(X) = F(X),$$

for $X \subseteq T'$. Then $F|_{T'}$ is an IndetermTreeSoft Set on T'.

Proof. The domain of $F|_{T'}$ is $P(T') \subseteq P(\text{Tree}(A))$. Any indeterminacy in the tree-structure (i) or codomain (ii) remains valid when considering a subtree. Moreover, if there exists $X_0 \subseteq T'$ so that $F(X_0)$ is indeterminate in the original definition, then $F|_{T'}(X_0) = F(X_0)$ is indeterminate as well. Thus $F|_{T'}$ satisfies the IndetermTreeSoft conditions. □

3.2 IndetermForestSoft Set

The IndetermForestSoft Set combines the concepts of IndetermSoft Sets and ForestSoft Sets. Its definitions and related details are provided below.

Definition 3.7 (IndetermForestSoft Set)**.** Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and let

$$\big\{F_t : P(\text{Tree}(A^{(t)})) \to P(H)\big\}_{t\in T}$$

be a finite or countable collection of TreeSoft Sets, each possibly containing some indeterminacy as described in Definition 3.1. Let

$$\text{Forest}\big(\{A^{(t)}\}\big) \;=\; \bigsqcup_{t\in T} \text{Tree}\big(A^{(t)}\big)$$

be the disjoint union of all attribute trees.

A mapping

$$\mathbf{F} : P\big(\text{Forest}(\{A^{(t)}\})\big) \;\longrightarrow\; P(H)$$

is called an *IndetermForestSoft Set* if for each $X \in P\big(\text{Forest}(\{A^{(t)}\})\big)$,

$$\mathbf{F}(X) \;=\; \bigcup_{\substack{t\in T \\ X\cap \text{Tree}(A^{(t)}) \neq \varnothing}} F_t\big(X \cap \text{Tree}(A^{(t)})\big),$$

and at least one of the following holds:

1. There is indeterminacy in some Tree$(A^{(t)})$ or in the way these trees are combined.
2. The codomain $P(H)$ has elements that are uncertain or ambiguous.
3. For some input X, the image $\mathbf{F}(X)$ is not uniquely defined or is subject to partial information.

In other words, an IndetermForestSoft Set unites multiple IndetermTreeSoft Sets (or TreeSoft Sets, some or all of which may have indeterminacy) into a single framework while preserving the possibility of ambiguity.

Example 3.8 (Cross-Department Candidate Shortlisting with Indeterminacy)**.** Let

$$U = \{\text{Alice, Bob, Charlie}\}, \quad H = U.$$

We have two departments $t = 1, 2$, with attribute trees:

$$\text{Tree}(A^{(1)}): \begin{cases} \text{Skills} = \{\text{Java, Python}\}, \\ \text{Experience} = \{\text{2–4yrs, 5+yrs}\}, \end{cases} \qquad \text{Tree}(A^{(2)}): \begin{cases} \text{Tools} = \{\text{SQL, R}\}, \\ \text{Certifications} = \{\text{None, Certified}\}. \end{cases}$$

Define two IndetermTreeSoft Sets

$$F_1 : P\big(\text{Tree}(A^{(1)})\big) \to P(H), \quad F_2 : P\big(\text{Tree}(A^{(2)})\big) \to P(H)$$

by the nonempty values

$$\begin{aligned} F_1(\{\text{Java}\}) &= \{\text{Alice, Bob}\}, \\ F_1(\{\text{Python}\}) &= \{\text{Bob, } \widetilde{\text{Charlie}}\}, \\ F_1(\{\text{Python, 5+yrs}\}) &= \{\text{Alice}\}, \\ F_2(\{\text{SQL}\}) &= \{\text{Alice, } \widetilde{\text{Charlie}}\}, \\ F_2(\{\text{Certified}\}) &= \{\text{Bob}\}, \\ F_2(\{\text{R, Certified}\}) &= \{\text{Alice}\}. \end{aligned}$$

Here $\widetilde{\text{Charlie}}$ indicates that Charlie's qualification is *indeterminate* under that criterion.

The forest of attributes is

$$\text{Forest}\big(\{A^{(1)}, A^{(2)}\}\big) = \text{Tree}(A^{(1)}) \bigsqcup \text{Tree}(A^{(2)}),$$

and the IndetermForestSoft Set

$$\mathbf{F} : P\big(\text{Forest}\big) \longrightarrow P(H)$$

is given by

$$\mathbf{F}(X) = \bigcup_{\substack{t=1,2 \\ X \cap \text{Tree}(A^{(t)}) \neq \varnothing}} F_t\big(X \cap \text{Tree}(A^{(t)})\big).$$

For instance, for

$$X = \{\text{Python, Certified}\},$$

we compute

$$\mathbf{F}(X) = F_1(\{\text{Python}\}) \cup F_2(\{\text{Certified}\}) = \{\text{Bob, } \widetilde{\text{Charlie}}\} \cup \{\text{Bob}\} = \{\text{Bob, } \widetilde{\text{Charlie}}\}.$$

Verification of indeterminacy:

(i) *Tree-structure indeterminacy.* In $\text{Tree}(A^{(1)})$, the node Python is uncertain for Charlie.

(ii) *Codomain indeterminacy.* The value $\mathbf{F}(X)$ contains $\widetilde{\text{Charlie}}$, an *ambiguous* membership in $P(H)$.

(iii) *Image indeterminacy.* For $X = \{\text{Python, Certified}\}$, the set $\mathbf{F}(X)$ is not uniquely determined regarding Charlie.

Hence $\mathbf{F}$ satisfies Definition 3.7 and models a real-world candidate shortlisting process across departments under uncertainty.

Theorem 3.9 (IndetermForestSoft Set as a Generalization of IndetermSoft and ForestSoft)**.** *Let U be a universe, $H \subseteq U$ non-empty, and let*

$$\big\{F_t : P(\mathrm{Tree}(A^{(t)})) \to P(H)\big\}_{t\in T}$$

be a (finite or countable) family of IndetermTreeSoft Sets. Let

$$\mathbf{F} : P\big(\bigsqcup_{t\in T} \mathrm{Tree}(A^{(t)})\big) \;\to\; P(H)$$

be their union as in Definition 3.7. Then:

1. *If each tree $\mathrm{Tree}(A^{(t)})$ has no sub-levels (so $\mathrm{Tree}(A^{(t)}) = A^{(t)}$) but retains any indeterminacy in its points or images, then $\mathbf{F}$ induces an IndetermSoft Set on the disjoint union of the attribute sets $\bigsqcup_{t\in T} A^{(t)}$.*
2. *If every F_t is fully determinate (no domain or codomain uncertainty) and the union introduces no new ambiguity, then $\mathbf{F}$ coincides with the classical ForestSoft Set obtained by uniting deterministic TreeSoft Sets.*

Proof. **(1) Reduction to IndetermSoft.** Assume each attribute tree satisfies $\mathrm{Tree}(A^{(t)}) = A^{(t)}$, a single-level set. Then

$$\mathrm{Forest}\big(\{A^{(t)}\}\big) = \bigsqcup_{t\in T} A^{(t)},$$

and

$$P\big(\mathrm{Forest}(\{A^{(t)}\})\big) = P\big(\bigsqcup_{t\in T} A^{(t)}\big).$$

Define the ground attribute set $A_{\mathrm{tot}} = \bigsqcup_{t\in T} A^{(t)}$, and a mapping

$$f : A_{\mathrm{tot}} \;\longrightarrow\; P(H), \qquad f(a) = \mathbf{F}(\{a\}).$$

Since $\mathbf{F}(\{a\})$ inherits any indeterminacy present in the corresponding F_t, at least one of the IndetermSoft conditions holds (uncertainty in the attribute a, in the image $f(a)$, or in the powerset $P(H)$). Thus (A_{tot}, f) is an IndetermSoft Set.

(2) Reduction to ForestSoft. Now suppose each F_t is a fully determinate TreeSoft Set and that no new ambiguity arises when taking the union. Then by definition of ForestSoft Set,

$$\mathbf{F}(X) = \bigcup_{t:\, X\cap \mathrm{Tree}(A^{(t)})\neq\varnothing} F_t\big(X \cap \mathrm{Tree}(A^{(t)})\big)$$

is exactly the union of deterministic values, with no indeterminacy in domain or codomain. Therefore $\mathbf{F}$ satisfies the axioms of a standard ForestSoft Set.

In both cases, $\mathbf{F}$ specializes to the corresponding classical structure, showing that IndetermForestSoft Sets indeed generalize both IndetermSoft and ForestSoft concepts. □

3.3 IndetermSuperHyperSoft Set

The IndetermSuperHyperSoft Set combines the concepts of SuperHyperSoft Sets and IndetermHyperSoft Sets. Its definition is provided below.

Definition 3.10 (IndetermSuperHyperSoft Set)**.** Let U be a universal set, and let $\{a_1, a_2, \ldots, a_n\}$ be n distinct attributes with corresponding sets of values A_i, each of which may exhibit partial or total indeterminacy. Define

$$\mathcal{C} \;=\; \mathcal{P}(A_1) \;\times\; \mathcal{P}(A_2) \;\times\; \cdots \;\times\; \mathcal{P}(A_n),$$

which forms the domain of a *SuperHyperSoft Set* (Definition [14]).

A mapping

$$F : \mathcal{C} \;\longrightarrow\; \mathcal{P}(U)$$

is called an *IndetermSuperHyperSoft Set* if one or more of the following indeterminacy conditions hold:

1. Some $\mathcal{P}(A_i)$ is uncertain or incompletely specified (i.e., the attribute values or their subsets are not fully determined).
2. The codomain $\mathcal{P}(U)$ contains partially unknown subsets or ambiguous memberships.
3. For at least one $\gamma \in \mathcal{C}$, the set $F(\gamma) \subseteq U$ is not uniquely defined or is subject to ambiguity.

Additionally, higher-level nesting (if each A_i itself is replaced by $\tilde{\mathcal{P}}(A_i)$ for extended hyperstructures) may introduce further levels of indeterminacy, consistent with the definition of IndetermHyperSoft Sets.

Example 3.11 (Scholarship Eligibility under Uncertain Criteria)**.** Let

$$U = \{\text{Alice, Bob, Charlie}\}, \qquad H = U.$$

Define two attributes:

$$A_1 = \{\text{High, Medium, Low}\}, \quad A_2 = \{\text{Science, Arts, Sports}\}.$$

Form the SuperHyperSoft domain

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2).$$

We define an IndetermSuperHyperSoft Set

$$F : \mathcal{C} \;\longrightarrow\; \mathcal{P}(U)$$

by specifying several values (using $\widetilde{\text{Bob}}$ to denote that Bob's eligibility is *indeterminate*):

$$\begin{aligned} F\big(\{\text{High}\}, \{\text{Science, Arts}\}\big) &= \{\text{Alice, } \widetilde{\text{Bob}}\}, \\ F\big(\{\text{Medium, Low}\}, \{\text{Sports}\}\big) &= \{\widetilde{\text{Charlie}}\}, \\ F\big(\{\text{High, Medium}\}, \{\text{Arts, Sports}\}\big) &= \{\text{Alice, Bob}\}. \end{aligned}$$

Verification of indeterminacy conditions:

(i) *Domain indeterminacy.* The set $\{\text{Science, Arts}\} \subseteq A_2$ is *uncertain*, since the distinction between Science and Arts criteria overlaps in some interdisciplinary cases.

(ii) *Codomain indeterminacy.* The images $F(\{\text{High}\}, \{\text{Science, Arts}\})$ and $F(\{\text{Medium, Low}\}, \{\text{Sports}\})$ include $\widetilde{\text{Bob}}$ and $\widetilde{\text{Charlie}}$, indicating *ambiguous* membership.

(iii) *Image indeterminacy.* For $\gamma = (\{\text{High}\}, \{\text{Science, Arts}\})$, the exact set of eligible students is not uniquely determined because Bob's interdisciplinary record is unclear.

Thus F satisfies the conditions of an IndetermSuperHyperSoft Set, modeling a real-world scholarship committee's decision process under multiple, interdependent uncertainties.

Theorem 3.12 (IndetermSuperHyperSoft Set Generalizes SuperHyperSoft and IndetermHyperSoft)**.** *Let U be a universal set, and let A_i $(1 \le i \le n)$ be attribute value sets, each possibly indeterminate. Set*

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n)$$

and let $F : \mathcal{C} \to \mathcal{P}(U)$ be an IndetermSuperHyperSoft Set as in Definition 3.10. Then:

1. *If every $\mathcal{P}(A_i)$ is fully determined and each $F(\gamma)$ is unambiguous, then F is precisely a SuperHyperSoft Set (no indeterminacy).*
2. *If $f : A_1 \times \cdots \times A_n \to \mathcal{P}(U)$ is an IndetermHyperSoft Set (with indeterminacy in the A_i, $\mathcal{P}(U)$, or in some $f(a_1, \ldots, a_n)$), then there is an embedding*

$$\widetilde{F} : \mathcal{C} \to \mathcal{P}(U), \qquad \widetilde{F}(\alpha_1, \ldots, \alpha_n) = \bigcup_{(a_1,\ldots,a_n) \in \alpha_1 \times \cdots \times \alpha_n} f(a_1, \ldots, a_n),$$

which is an IndetermSuperHyperSoft Set extending f.

Proof. **(1) Determinate case.** Suppose each $\mathcal{P}(A_i)$ is a classical power set (no ambiguity) and that for every $\gamma \in \mathcal{C}$, $F(\gamma) \subseteq U$ is uniquely specified. Then none of the indeterminacy conditions in Definition 3.10 applies, so F meets exactly the axioms of a SuperHyperSoft Set (see [14]). Hence F is a SuperHyperSoft Set.

(2) Embedding IndetermHyperSoft into IndetermSuperHyperSoft. Let

$$f : A_1 \times \cdots \times A_n \longrightarrow \mathcal{P}(U)$$

be an IndetermHyperSoft Set, so at least one of:

- some A_i or its subsets are indeterminate,
- some $f(a_1, \ldots, a_n)$ is ambiguous,
- $\mathcal{P}(U)$ contains uncertain subsets,

holds. Define

$$\widetilde{F}(\alpha_1, \ldots, \alpha_n) = \bigcup_{(a_1,\ldots,a_n) \in \alpha_1 \times \cdots \times \alpha_n} f(a_1, \ldots, a_n), \quad (\alpha_1, \ldots, \alpha_n) \in \mathcal{C}.$$

Then:

- If any $\alpha_i \subseteq A_i$ or any $f(a_1, \ldots, a_n)$ is indeterminate, the union remains indeterminate, so $\widetilde{F}$ satisfies the corresponding condition in Definition 3.10.
- $\widetilde{F}$ extends f because whenever each $\alpha_i = \{a_i\}$, we have

$$\widetilde{F}(\{a_1\}, \ldots, \{a_n\}) = f(a_1, \ldots, a_n).$$

Thus $\widetilde{F}$ is an IndetermSuperHyperSoft Set that coincides with the given IndetermHyperSoft Set on singleton inputs, demonstrating that every IndetermHyperSoft Set embeds into the IndetermSuperHyperSoft framework. □

Theorem 3.13 (Closure under Union)**.** *Let U be a universe and $\mathcal{C} = \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n)$. Suppose*

$$F,\ G : \mathcal{C} \longrightarrow \mathcal{P}(U)$$

are two IndetermSuperHyperSoft Sets. Define

$$(F \vee G)(\gamma) \ = \ F(\gamma) \ \cup \ G(\gamma), \qquad \forall\, \gamma \in \mathcal{C}.$$

Then $F \vee G$ is again an IndetermSuperHyperSoft Set.

Proof. We must show that $F \vee G$ satisfies at least one of the indeterminacy conditions in Definition 3.10.

(i) Domain indeterminacy. If any $\mathcal{P}(A_i)$ is uncertain, this is independent of the choice of F or G, hence remains true for $F \vee G$.

(ii) Codomain indeterminacy. If $\mathcal{P}(U)$ admits ambiguous subsets, then since $F(\gamma)$ or $G(\gamma)$ may be ambiguous, their union $F(\gamma) \cup G(\gamma)$ is likewise ambiguous.

(iii) Image indeterminacy. Suppose there exists $\gamma_0 \in \mathcal{C}$ such that $F(\gamma_0)$ is ambiguous or $G(\gamma_0)$ is ambiguous. Then

$$(F \vee G)(\gamma_0) = F(\gamma_0) \cup G(\gamma_0)$$

remains ambiguous, because the union of a non-unique set with any other preserves non-uniqueness.

At least one of (i)–(iii) holds for $F \vee G$, so by Definition 3.10, $F \vee G$ is an IndetermSuperHyperSoft Set. □

Theorem 3.14 (Closure under Intersection)**.** *Under the same hypotheses as Theorem 3.13, define*

$$(F \wedge G)(\gamma) \;=\; F(\gamma) \;\cap\; G(\gamma).$$

Then $F \wedge G$ is an IndetermSuperHyperSoft Set.

Proof. Again let $H = F \wedge G$. Conditions (i) and (ii) from Definition 3.10 carry over unchanged. For (iii), if there exists γ_0 with $F(\gamma_0)$ or $G(\gamma_0)$ ambiguous, then

$$H(\gamma_0) = F(\gamma_0) \cap G(\gamma_0)$$

remains ambiguous, since intersecting a non-unique set with any other yields non-uniqueness. Hence H satisfies at least one indeterminacy condition and is an IndetermSuperHyperSoft Set. □

Theorem 3.15 (Projection onto a Subset of Attributes)**.** *Let F be an IndetermSuperHyperSoft Set on $\mathcal{C} = \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n)$. Fix a nonempty index set $I \subseteq \{1, \ldots, n\}$ and set*

$$\mathcal{C}_I = \prod_{i \in I} \mathcal{P}(A_i).$$

Define the projection

$$F_I : \mathcal{C}_I \;\longrightarrow\; \mathcal{P}(U), \qquad F_I\big((\alpha_i)_{i \in I}\big) = \bigcup_{\substack{(\beta_1, \ldots, \beta_n) \in \mathcal{C} \\ \beta_i = \alpha_i \ (i \in I)}} F(\beta_1, \ldots, \beta_n).$$

Then F_I is an IndetermSuperHyperSoft Set on $\mathcal{C}_I$.

Proof. We verify the indeterminacy conditions for F_I:

(i) Domain indeterminacy. Any uncertainty in the original domains $\mathcal{P}(A_i)$ for $i \in I$ persists in the projected domains $\mathcal{P}(A_i)$, so F_I inherits domain-level indeterminacy.

(ii) Codomain indeterminacy. Since $F(\gamma)$ may be ambiguous for some γ, their union over any fiber of the projection remains ambiguous in $\mathcal{P}(U)$.

(iii) Image indeterminacy. If there exists $\delta \in \mathcal{C}_I$ and some extension $\gamma \in \mathcal{C}$ with $\gamma|_I = \delta$ for which $F(\gamma)$ is ambiguous, then

$$F_I(\delta) = \bigcup_{\gamma|_I = \delta} F(\gamma)$$

is ambiguous as well (the union of a non-unique set preserves non-uniqueness).

Thus F_I satisfies at least one indeterminacy condition and is an IndetermSuperHyperSoft Set on $\mathcal{C}_I$. □

3.4 Incomplete HyperSoft Set

The Incomplete HyperSoft Set is a generalized concept derived from the Incomplete Soft Set. Its formal definition is provided below.

Definition 3.16 (Incomplete HyperSoft Set)**.** Let U be a universal set, and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be m (possibly indeterminate) attribute domains. Define the Cartesian product

$$\mathcal{C} \;=\; \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m.$$

An *Incomplete HyperSoft Set* over U is defined as a tuple

$$\Big(U,\ \mathcal{A}_1,\ \mathcal{A}_2,\ \ldots,\ \mathcal{A}_m,\ G\Big),$$

where $G : \mathcal{C} \to \big(\mathcal{P}(U) \cup \{*\}\big)$. Concretely,

$$G(\gamma) \;=\; \begin{cases} S_\gamma \subseteq U, & \text{if sufficient information is available,} \\ *, & \text{if the information is missing or undefined,} \end{cases}$$

for each $\gamma \in \mathcal{C}$. Here, $*$ denotes incomplete or missing data in analogy with the incomplete soft set.

In this definition:

1. U is a non-empty universe of discourse.
2. $\mathcal{A}_i$ are attribute domains, each of which may be partially incomplete or fully defined.
3. $\mathcal{C} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$ represents all possible combinations of attribute values.
4. $G(\gamma)$ lies in $\mathcal{P}(U) \cup \{*\}$. If $G(\gamma) = *$, the mapping is incomplete for the given γ.

Example 3.17 (Clothing Purchase Data with Incomplete Records)**.** Let

$$U = \{\text{Alice, Bob, Charlie}\}$$

be the set of customers. Consider two product attributes:

$$\mathcal{A}_1 = \{\text{Red, Blue}\}, \quad \mathcal{A}_2 = \{\text{Small, Medium, Large}\}.$$

The Cartesian product

$$\mathcal{C} \;=\; \mathcal{A}_1 \times \mathcal{A}_2 \;=\; \{\,(c, s) \mid c \in \{\text{Red}, \text{Blue}\},\ s \in \{\text{Small}, \text{Medium}, \text{Large}\}\}$$

indexes all color-size variants.

Define

$$G : \mathcal{C} \;\longrightarrow\; \mathcal{P}(U) \cup \{*\}$$

by the following table:

$$\begin{aligned} G(\text{Red}, \text{Small}) &= \{\text{Alice, Bob}\}, \\ G(\text{Red}, \text{Medium}) &= *, \\ G(\text{Red}, \text{Large}) &= \{\text{Charlie}\}, \\ G(\text{Blue}, \text{Small}) &= \{\text{Bob}\}, \\ G(\text{Blue}, \text{Medium}) &= \{\text{Alice}\}, \\ G(\text{Blue}, \text{Large}) &= *. \end{aligned}$$

Here, $*$ denotes that the sales data for that specific color-size combination is missing or was not recorded.

Then

$$\big(U,\ \mathcal{A}_1,\ \mathcal{A}_2,\ G\big)$$

is an *Incomplete HyperSoft Set* over U. It models customer purchase behavior under incomplete data: for some variants (e.g. Red–Medium and Blue–Large), the mapping G returns $*$, indicating unavailable information.

Theorem 3.18. *Every Incomplete HyperSoft Set is an IndetermHyperSoft Set, and the Incomplete Soft Set arises as the single-attribute special case.*

Proof. Let

$$\big(U,\ \mathcal{A}_1, \ldots, \mathcal{A}_m,\ G\big)$$

be an Incomplete HyperSoft Set as in Definition 3.16, with $\mathcal{C} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$.

(1) Embedding into IndetermHyperSoft. Recall (Definition of IndetermHyperSoft Set) that a pair $(F, \mathcal{C})$ is an *IndetermHyperSoft Set* if

$$F : \mathcal{C} \ \longrightarrow\ \mathcal{P}(U)$$

is allowed to take *indeterminate* values in any of the following ways:

- one or more domains $\mathcal{A}_i$ are themselves uncertain or incomplete,
- the codomain $\mathcal{P}(U)$ admits fuzzy or ambiguous subsets,
- there exists $\gamma \in \mathcal{C}$ with $F(\gamma)$ not uniquely determined.

Define

$$F : \mathcal{C} \ \longrightarrow\ \mathcal{P}(U) \cup \{*\}, \qquad F(\gamma) = G(\gamma).$$

Then each occurrence $G(\gamma) = *$ is exactly an *indeterminate image* in the sense of IndetermHyperSoft Sets. No further restrictions are imposed on the domains $\mathcal{A}_i$. Hence $(F, \mathcal{C})$ satisfies the definition of an IndetermHyperSoft Set.

(2) Reduction to Incomplete Soft Set. Let (U, E, f) be an Incomplete Soft Set with

$$f : E \ \longrightarrow\ \mathcal{P}(U) \cup \{*\}.$$

Take $m = 1$ and $\mathcal{A}_1 = E$. Then $\mathcal{C} = \mathcal{A}_1 = E$, and setting $G = f$ yields an Incomplete HyperSoft Set

$$\big(U, E,\ G = f\big).$$

Thus every Incomplete Soft Set is exactly the single-attribute case of an Incomplete HyperSoft Set. □

Theorem 3.19 (Closure under Union)**.** *Let*

$$\big(U,\ \mathcal{A}_1, \ldots, \mathcal{A}_m,\ G_1\big) \quad \textit{and} \quad \big(U,\ \mathcal{A}_1, \ldots, \mathcal{A}_m,\ G_2\big)$$

be two Incomplete HyperSoft Sets with the same attribute domains and $\mathcal{C} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$*. Define*

$$G(\gamma) = \begin{cases} G_1(\gamma) \cup G_2(\gamma), & G_1(\gamma) \neq *,\ G_2(\gamma) \neq *, \\ G_1(\gamma), & G_1(\gamma) \neq *,\ G_2(\gamma) = *, \\ G_2(\gamma), & G_1(\gamma) = *,\ G_2(\gamma) \neq *, \\ *, & G_1(\gamma) = *,\ G_2(\gamma) = *. \end{cases}$$

Then $\big(U, \mathcal{A}_1, \ldots, \mathcal{A}_m, G\big)$ *is an Incomplete HyperSoft Set.*

Proof. We must show $G : \mathcal{C} \to \mathcal{P}(U) \cup \{*\}$ satisfies Definition 3.16.

(Domain indeterminacy) Any incompleteness in the domains $\mathcal{A}_i$ remains unchanged by forming unions, so condition (1) of Definition 3.16 persists.

(Codomain indeterminacy) If $\mathcal{P}(U)$ admits ambiguous or partial subsets, then whenever $G_1(\gamma)$ or $G_2(\gamma)$ is a non-unique subset, their union is also non-unique, satisfying condition (2).

(Image indeterminacy) Take any $\gamma_0 \in \mathcal{C}$. If $G_1(\gamma_0) = *$ or $G_2(\gamma_0) = *$, then by construction $G(\gamma_0) = *$. Otherwise $G_1(\gamma_0)$ or $G_2(\gamma_0)$ may be ambiguous, and so $G(\gamma_0) = G_1(\gamma_0) \cup G_2(\gamma_0)$ is ambiguous, satisfying condition (3).

Thus G meets at least one indeterminacy condition for each γ, and is an Incomplete HyperSoft Set. □

Theorem 3.20 (Closure under Intersection)**.** *With the same setup as Theorem 3.19, define*

$$G'(\gamma) = \begin{cases} G_1(\gamma) \cap G_2(\gamma), & G_1(\gamma) \neq *,\ G_2(\gamma) \neq *, \\ *, & \textit{otherwise.} \end{cases}$$

Then $\big(U, \mathcal{A}_1, \ldots, \mathcal{A}_m, G'\big)$ *is an Incomplete HyperSoft Set.*

Proof. Again, domain indeterminacy (1) and codomain indeterminacy (2) carry over from G_1, G_2. For image indeterminacy (3), if either $G_1(\gamma)$ or $G_2(\gamma)$ equals $*$, then $G'(\gamma) = *$. If both are non-$*$ but at least one is ambiguous, their intersection remains ambiguous. Hence G' fulfills Definition 3.16. □

Theorem 3.21 (Projection onto a Subset of Attributes)**.** *Let* $\big(U, \mathcal{A}_1, \ldots, \mathcal{A}_m, G\big)$ *be an Incomplete HyperSoft Set on* $\mathcal{C} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$*. Fix a nonempty index set* $I \subseteq \{1, \ldots, m\}$ *and let* $\mathcal{C}_I = \prod_{i \in I} \mathcal{A}_i$*. Define the* projection

$$G_I(\alpha) = \begin{cases} \bigcup\limits_{\substack{\gamma \in \mathcal{C} \\ \gamma|_I = \alpha}} G(\gamma), & \textit{if } G(\gamma) \neq * \textit{ for all such } \gamma, \\ *, & \textit{otherwise,} \end{cases}$$

for each $\alpha \in \mathcal{C}_I$*. Then* $\big(U, \{\mathcal{A}_i\}_{i \in I}, G_I\big)$ *is an Incomplete HyperSoft Set on* $\mathcal{C}_I$*.*

Proof. **(1) Domain indeterminacy.** Any incompleteness in the original domains $\mathcal{A}_i$ for $i \in I$ remains in $\mathcal{C}_I$.

(2) Codomain indeterminacy. If $\mathcal{P}(U)$ is ambiguous, then unions of its elements remain ambiguous.

(3) Image indeterminacy. For a given $\alpha \in \mathcal{C}_I$, if any extension γ with $\gamma|_I = \alpha$ has $G(\gamma) = *$, then $G_I(\alpha) = *$. Otherwise, the union over a family of possibly ambiguous subsets is ambiguous.

Thus G_I satisfies at least one indeterminacy condition for each α, making it an Incomplete HyperSoft Set. □

3.5 Incomplete SuperHyperSoft Set

The Incomplete SuperHyperSoft Set is a generalized concept derived from the Incomplete HyperSoft Set. Its formal definition is provided below.

Definition 3.22 (Incomplete SuperHyperSoft Set)**.** Let U be a universal set, and let $\{a_1, a_2, \ldots, a_n\}$ be $n \geq 1$ distinct attributes, each with a set of possible values A_i. Define

$$\mathcal{C} \;=\; \mathcal{P}(A_1) \;\times\; \mathcal{P}(A_2) \;\times\; \cdots \;\times\; \mathcal{P}(A_n),$$

as in the definition of a *SuperHyperSoft Set*. An *Incomplete SuperHyperSoft Set* is a tuple

$$\Big(U,\ \{A_i\}_{i=1}^{n},\ F\Big),$$

where

$$F : \mathcal{C} \;\longrightarrow\; \big(\mathcal{P}(U) \cup \{*\}\big).$$

Specifically, for each $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i \subseteq A_i$,

$$F(\gamma) \;=\; \begin{cases} S_\gamma \subseteq U, & \text{if the information is complete,} \\ *, & \text{if the information is incomplete or undefined.} \end{cases}$$

Here, $*$ indicates a missing or indeterminate value at the superhyper level (e.g., incomplete data about chosen subsets α_i or how they map into U).

Example 3.23 (Smartphone Models under Incomplete SuperHyperSoft Criteria)**.** Let

$$U = \{\text{ModelX, ModelY, ModelZ}\}$$

be the set of available smartphone models. Consider three attribute domains:

$$A_1 = \{\text{5G, NFC}\}, \quad A_2 = \{\text{Black, White}\}, \quad A_3 = \{\text{64GB, 128GB}\}.$$

Form the SuperHyperSoft domain

$$\mathcal{C} = \mathcal{P}(A_1) \;\times\; \mathcal{P}(A_2) \;\times\; \mathcal{P}(A_3).$$

Define

$$F : \mathcal{C} \;\longrightarrow\; \mathcal{P}(U) \cup \{*\}$$

by specifying its values on several representative points (where "$*$" indicates missing or uncollected data):

$$\begin{aligned} F\big(\{\text{5G}\}, \{\text{Black}\}, \{\text{64GB}\}\big) &= \{\text{ModelX, ModelY}\}, \\ F\big(\{\text{NFC}\}, \{\text{White}\}, \{\text{128GB}\}\big) &= \{\text{ModelZ}\}, \\ F\big(\{\text{5G, NFC}\}, \{\text{White}\}, \{\text{128GB}\}\big) &= *, \\ F\big(\varnothing, \{\text{Black, White}\}, \{\text{64GB, 128GB}\}\big) &= \{\text{ModelX, ModelZ}\}. \end{aligned}$$

Explanation of indeterminacy:

- *Domain-level incompleteness:* For the combination $(\{\text{5G, NFC}\}, \{\text{White}\}, \{\text{128GB}\})$, customer preference data were not gathered, so $F = *$.
- *Image-level incompleteness:* Even when a subset of attributes is specified, the corresponding models may be under review (e.g. ModelY's 128GB sensor performance), yielding $*$.
- *Complete entries:* Other entries, such as $(\{\text{5G}\}, \{\text{Black}\}, \{\text{64GB}\})$, have full data and map to concrete subsets of U.

Thus $\big(U, \{A_i\}, F\big)$ is an *Incomplete SuperHyperSoft Set*, modeling a real-world scenario where multi-attribute purchase data may be partially missing or undefined.

Theorem 3.24. *An Incomplete SuperHyperSoft Set is a special case of an IndetermHyperSoft Set (with superhyper structure) and also extends the concept of a SuperHyperSoft Set to accommodate incomplete information.*

Proof. Let

$$\big(U, \{A_i\}_{i=1}^n, F\big)$$

be an Incomplete SuperHyperSoft Set as in Definition 3.22, with

$$\mathcal{C} = \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n).$$

(1) It is an IndetermHyperSoft Set with superhyper-level indeterminacy. Recall (Definition of IndetermHyperSoft Set) that an IndetermHyperSoft Set over U with domain $\mathcal{C}$ is any mapping

$$F' : \mathcal{C} \;\longrightarrow\; \mathcal{P}(U)$$

which is allowed to exhibit indeterminacy in any of the following ways:

(a) Some attribute domain $\mathcal{P}(A_i)$ is itself uncertain or partially specified.

(b) The codomain $\mathcal{P}(U)$ admits ambiguous or fuzzy subsets.

(c) There exists $\gamma \in \mathcal{C}$ for which $F'(\gamma)$ is not uniquely determined.

In our Incomplete SuperHyperSoft Set, the map

$$F(\gamma) \ \in \ \mathcal{P}(U) \cup \{*\}$$

explicitly returns $*$ whenever the information for γ is missing or undefined. Viewing $*$ as "indeterminate output," we see that each occurrence of $F(\gamma) = *$ satisfies condition (c) above. Moreover, if any $\mathcal{P}(A_i)$ is incomplete or if $\mathcal{P}(U)$ contains ambiguous subsets, then conditions (a) or (b) are also met. Hence F meets the definition of an IndetermHyperSoft Set on the same superhyper domain $\mathcal{C}$.

(2) It extends the SuperHyperSoft Set by allowing $*$. A SuperHyperSoft Set over U with domain $\mathcal{C}$ is precisely a mapping

$$F_0 : \mathcal{C} \ \longrightarrow \ \mathcal{P}(U),$$

with no undefined values. The Incomplete SuperHyperSoft Set broadens this by permitting $* \in \mathcal{P}(U) \cup \{*\}$ whenever the data are missing. Concretely, if one restricts to those $\gamma \in \mathcal{C}$ for which $F(\gamma) \neq *$, then

$$F|_{\{\gamma : F(\gamma) \neq *\}} \ : \ \mathcal{C} \ \longrightarrow \ \mathcal{P}(U)$$

coincides with the original SuperHyperSoft mapping. Thus, in the absence of any $*$-values, F reduces exactly to a SuperHyperSoft Set.

Combining (1) and (2), we conclude that every Incomplete SuperHyperSoft Set is simultaneously

- a special case of an IndetermHyperSoft Set (with nesting up to superhyper level, and indeterminacy encoded by $*$), and
- a strict generalization of a SuperHyperSoft Set (recovering it when no information is missing).

This establishes the desired generalization. □

4 Conclusion

In this paper, we proposed three novel extensions—*IndetermSuperHyperSoft Sets*, *IndetermForestSoft Sets*, and *IndetermTreeSoft Sets*—providing rigorous definitions, illustrative examples, and a comprehensive study of how these structures extend and unify existing soft–set models.

In the future, we hope that further research will address extensions of these frameworks to Graphs [40,41], HyperGraphs [42–44], and SuperHyperGraphs [45–47], as well as to QuantumSoft Sets [48–50]. We also anticipate progress in the development of algorithms and practical case studies related to these models.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors affirm that, to the best of their knowledge, this manuscript represents their original research. It has not been previously published in any journal, nor is it currently being considered for publication elsewhere.

Disclaimer on Computational Tools

No computer-based tools—such as symbolic computation systems, automated theorem provers, or proof assistants (e.g., Mathematica, SageMath, Coq)—were employed in the development, analysis, or verification of the results contained in this paper. All derivations and proofs were conducted manually through analytical methods by the authors.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Consent to Participate

Not applicable.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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Chapter 3

Unveiling the n-SuperHyperGraph: A Comprehensive Overview of Graph Evolution

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Abstract

The **n-SuperHyperGraph** emerges as the most encompassing and flexible representation, extending beyond traditional graph structures. The introduction of **SuperHyperGraphs (SHGs)**—a generalization of traditional hypergraphs allows for more complex and nuanced models of systems with interconnected vertices. This chapter explores the foundational concepts, mathematical framework, and practical implications of the n-SuperHyper Graph and **SuperHyperGraphs-based domination number**, detailing its real-world applications across various domains, where the goal is to identify the minimal set of vertices that can effectively dominate or control a system. Super hypergraphs enhance traditional hypergraphs by capturing higher-order interactions and complex relationships, offering a richer representation of data. This framework improves neural network architectures by modeling multi-way dependencies and enabling better generalization. Through theoretical and empirical analysis, it showed how super hypergraphs can advance tasks in relational data, graph learning, and multi-modal inputs for smarter systems.

Keywords : SuperHyperGraphs, domination number, neural network, complex relationships.

1. Introduction

Graphs, the most fundamental data structures in mathematics and computer science, consist of vertices (nodes) connected by edges (links). Simple graphs have edges between pairs of vertices, while **hypergraphs** ([1], [4], [6]) allow edges to connect multiple vertices simultaneously. A hyperedge e in a hypergraph H connects k vertices v_1, v_2, ... , v_k where k can be any integer greater than 2. This enhancement allows for more expressive models, such as workflows, collaborations, or resource-sharing networks, where interactions involve groups rather than pairs. These structures have been widely used to represent various systems, such as social networks, transportation systems, biological networks, and knowledge graphs. However, traditional graph models have limitations in capturing intricate relationships among entities.

In the world of mathematics, the concept of set theory provides the foundation for a variety of mathematical structures. One such fascinating extension of set theory is the Super Hyper Soft Set(SHSS). This mathematical structure broadens the traditional soft set theory, allowing for greater flexibility in handling uncertain, incomplete, and dynamic information.

SHSS integrate the concept of hyper sets with soft set theory. Hyper sets allow the inclusion of elements from different types, while soft sets introduce parameters that can vary with different conditions or criteria. This explores the foundational concepts, operations, and structures related to SHSS and extend these ideas to the formation of Super Hyper Graphs ([3], [5]).

Despite their expressive power, hypergraphs have limitations when it comes to capturing relationships across multiple dimensions or integrating information from diverse domains. They struggle to represent complex, dynamic systems where interactions can span more than simple pairwise relationships. The n-SuperHyperGraph [6] (n-SHG) addresses these limitations by expanding hypergraph theory into a multi-dimensional framework.

The **domination number** in traditional graphs is a fundamental concept where vertices are selected such that every other vertex in the graph is adjacent to at least one vertex in the set. The goal is to minimize the size of this set while ensuring complete control over the graph. Traditional graphs focus on pairwise interactions between vertices and edges, which can be sufficient for many real-world applications but falls short when modeling systems with more complex relationships.

The main challenge with traditional domination is that it fails to capture interactions that involve multiple vertices simultaneously. Many real-world systems such as social networks, biological systems, and communication networks exhibit relationships where more than two vertices are interconnected. This is where SuperHyperGraphs come into play, providing an expanded framework that allows edges to connect more than two vertices.

Super hypergraphs represent an extension of traditional hypergraphs, where nodes still represent entities, but the hyperedges capture higher-order interactions and more complex relationships between them. In contrast to regular graphs, which only model pairwise relationships, super hypergraphs can encode multi-way dependencies, enabling a more expressive and nuanced representation of data. This feature makes super hypergraphs particularly suitable for tasks requiring the modeling of complex structures, such as graph learning, relational data, and multi-

modal inputs. Neural networks have traditionally struggled to fully capture such intricate relationships, often relying on oversimplified assumptions. By incorporating super hypergraphs, neural networks can gain a deeper understanding of data, improving their performance and generalization capabilities. The ability to model these higher-order dependencies offers a powerful tool for tasks ranging from recommendation systems to social network analysis. Super hypergraphs also introduce new opportunities for better efficiency in training deep learning models. With these advancements, the field of neural networks can benefit from more flexible, scalable architectures. This explores the potential of integrating super hypergraphs into neural network design, showcasing how this novel framework can address challenges in various machine learning domains.

2. Basic Concepts

Power set: The powerset of a set S, denoted P (S), is the set of all subsets of S, including the empty set and S itself. Formally, P (S) = $\{A \mid A \subseteq S\}$.

***n*-th power set:** The n-th powerset of H, denoted Pn (H), is defined recursively as,

$$P_1(H) = (H),\ P_{n+1}(H) = (P_n(H)) \text{ for } n \geq 1.$$

Similarly, the n-th non-empty powerset of H, denoted P^*_n (H), is defined as:$_1{}^*$ (H) = $P*$ (H), P^*_{n+1} (H) = $P*$ (P^*_n (H)).

Result: A n-th powerset is a generalized concept of a powerset.

Super Hyper Soft Set: A Super Hyper Soft Set (SHSS) is a mathematical structure that is characterized by:

- A **Universal Set** U, which consists of all possible objects or elements.
- A **Parameter Set** P, which contains parameters that define different conditions or criteria.
- A **Function** f:U×P→V, where V is a set of membership degrees or classifications, often real-valued.

A Super Hyper Soft Set S can be represented as:

S={(x, f(x, p))|x∈ U , p ∈ P} where f(x, p) denotes the membership degree of element x in the condition p.

Operations on Super Hyper Soft Sets

The basic operations for SHSS include:

- **Union**: $S \cup T = \{(x, \max(f(x, p), g(x, p)))|x \in U, p \in P\}$
- **Intersection**: $S \cap T = \{(x, \min(f(x, p), g(x, p)))|x \in U, p \in P\}$
- **Complement**: $S^c = \{(x, 1 - f(x,p))|x \in U, p \in P\}$ Where S and T are SHSS, and f, g are membership functions.

2.1. Super Hyper Graphs from Super Hyper Soft Sets

A Super Hyper Graph (SHG) [1]is an extension of a graph in which vertices and edges can belong to multiple sets simultaneously, integrating concepts from hyper graphs, soft graphs, and fuzzy graphs.

A Super Hyper Graph G can be defined as: $G = (V, E, \phi)$ where

- V is the set of vertices (also elements of a universal set U).
- E is the set of edges, where an edge E can have memberships in several super hyper soft sets.
- $\phi: E \times P \rightarrow V$ represents the membership function for edges in various conditions.

2.2. Characteristics of Super Hyper Graphs

- **Multiset of Edges**: Each edge $e \in E$ belongs to a family of soft sets, allowing it to be an element of multiple substructures within the super hyper graph.
- **Membership Function**: ϕ (e, p) gives the degree to which the edge e belongs to condition p, which can vary between 0 and 1.

2.2.1. Construction Techniques

Building Super Hyper Soft Sets

To construct a Super Hyper Soft Set,

- A universal set U containing all possible elements.
- A parameter set P defining various conditions or scenarios.
- A membership function $f : U \times P \rightarrow [0,1]$ that assigns membership values.

Transitioning to Super Hyper Graphs

- Convert Super Hyper Soft Sets [4] into Super Hyper Graphs by defining edges that include memberships across different sets.
- Vertices represent elements in the universal set, while edges represent relationships with varying strengths across conditions.

2.3 The Concept of SuperHyperGraphs and Domination Number

A SuperHyperGraph (SHG) extends beyond traditional hypergraphs by enabling edges that can connect multiple sets of vertices across different dimensions. In such a graph, the domination number can be defined not just in terms of individual vertices but also in terms of superedges that link multiple vertices.

Domination in SuperHyperGraphs

Given a Super Hyper Graph G = (V, E):

- **Vertices**: V is the set of nodes.
- **Edges**: E represents the set of superedges, where each edge can connect multiple vertices simultaneously.

The SuperHyperGraph domination number γ_{SHG} is defined as the minimum number of vertices (or sets) required to ensure that every vertex in the graph is connected by at least one edge. In SHGs, domination is more comprehensive, as it considers connections between groups of vertices rather than just individual nodes.

2.3.1 Representation of Domination in SHGs

Mathematically, for a SuperHyperGraph G=(V,E) domination can be represented as,

$$\gamma_{SHG} = \min |D| \text{ such that } \forall v \in V, \exists e \in E \text{ with } v \in e$$

where D is the dominating set, and |D| represents the size of the set.

Theorem 2.1: Every superhypergraph can be transformed into a hypergraph.

Proof:

Let H = (V, E) be a superhypergraph, where:

• V ⊆ P (U) for some underlying set U, meaning each v ∈ V is either a single element of U or a subset of U (termed a supervertex).

• E ⊆ P (V), with each e ∈ E linking subsets of elements in V (known as superedges).

Construct a hypergraph H′ = (V ′, E′) that is equivalent to H but adheres to the structure of a hypergraph.

Construction of H′: Define V ′ as the union of all elements in the supervertices of V ,

Specifically: $V' = \bigcup_{v \in V} v$

Here, each element in V ′ is a unique element of U, effectively “flattening” the supervertices of V into single elements in V ′.

Define the edge set E′ as follows: for each superedge $e = \{v_1, v_2, \ldots, v_n\} \in E$ (where $v_i \in V$), construct an edge $e' \subseteq V'$ by taking the union of the elements within each supervertex

v_i, i.e., $e' = \bigcup_{v \in E} v$

The construction ensures that each $e' \subseteq V'$, thus meets the definition of a hyperedge in H'.

Verification: The transformed structure $H' = (V', E')$ satisfies the following properties of a hypergraph:

• V' is a set of vertices, where each element corresponds to an element in the underlying set U (not subsets of U), ensuring $V' \subseteq U$.

• Each edge $e' \in E'$ is a subset of V' and does not contain supervertices, adhering to the standard hypergraph structure where each edge is a subset of the vertex set.

Thus, the transformation from $H = (V, E)$ to $H' = (V', E')$ is valid, and H' is a hypergraph as required.

3. Example of Superhypergraph

Demonstration on how a superhypergraph, which includes higher-level structures, can be simplified into a hypergraph by removing those higher-level relationships.

Consider a social media platform where users are not only connected directly through posts (edges) but also through group activities (higher-order connections).

- Users are represented as **vertices**.
- A hyperedge in the superhypergraph can represent a post made by multiple users, where each user is part of the hyperedge.
- But now, imagine a situation where groups of users are also forming groups to interact around certain topics. These groups themselves can have hyperedges connecting them. In the superhypergraph, you might have a higher-order relationship that connects multiple groups, creating a more complex structure.

3.1 Transforming into a Hypergraph

To transform this superhypergraph into a normal hypergraph**,** The structure by reducing the complexity of higher-level relationships are simplified

1. Remove the higher-order relationships**:** Instead of groups of users being connected to other groups, we focus only on the direct connections between users (the basic vertices).

2. Simplify the hyperedges**:** Instead of having hyperedges that represent sets of groups, we would represent each group activity directly as a hyperedge connecting only the users involved in that activity, without considering the groups themselves as higher-order structures.

3.1.1 Example

- A Superhypergraph would involve users A, B, CA, B, CA, B, C, and D being part of posts, and posts themselves might be grouped under larger categories like "Sports", "Technology", etc., where each category can connect with other categories in a higher-level way.
- A Hypergraph would simplify this by saying that:
    - A hyperedge connects A, B, CA, B, CA, B, C (a post involving these users).
    - Another hyperedge connects B, C, DB, C, DB, C, D (another post involving these users).

By doing so, remove the more abstract relationships between groups, making the structure simpler and closer to the traditional hypergraph. This is how a superhypergraph can be transformed into a hypergraph.

The key difference lies in the abstraction level superhypergraphs [3] allow for connections that involve multiple layers (e.g., groups of users), while hypergraphs focus on direct connections between entities (just the users or vertices). The transformation process involves simplifying the structure by collapsing higher-order connections.

Example Scenario:

Step 1: Superhypergraph

Let's consider a superhypergraph that models relationships in a **university** setting:

- **Vertices (V)**: The set of students in the university.

  $V=\{S_1, S_2, S_3, S_4, S_5\}$

  where:
    - S_1 is Alice,
    - S_2 is Bob,
    - S_3 is Carol,
    - S_4 is Dave,

- S_5 is Eve.

- **Hyperedges (H)**: Define some **hyperedges** in the superhypergraph:

(i) First-level Hyperedge (Posts): Each post made by students can be seen as a hyperedge:

$H1=\{S_1,S_2\}$(Post 1 made by Alice and Bob)

$H2=\{S_3,S_4,S_5\}$(Post 2 made by Carol, Dave, and Eve)

(ii) Second-level Hyperedges (Groups): The posts themselves can be grouped under topics (a higher-order structure):

- $G_1=\{H_1, H_2\}$(Group 1 - "Technology posts")
- $G_2=\{H_1\}$(Group 2 - "Social posts") Here, the sets of posts (like H_1 and H_2) form new hyperedges that connect different groups of posts under a shared topic.

In this superhypergraph:

- $V=\{S_1,S_2,S_3,S_4,S_5\}$ is the set of vertices (students).
- The first-level hyperedges H1 and H2 connect students based on their posts.
- The second-level hyperedges G_1 and G_2 connect groups of posts into broader categories, forming a higher-order structure.

Step 2: Transforming the Superhypergraph into a Hypergraph

To transform this superhypergraph into a regular hypergraph, we need to **simplify** the structure by eliminating the higher-order relationships (i.e., the second-level hyperedges that group posts together).

1. **Vertices (V)**: We keep the same set of vertices, which represent students:
   $V= \{S_1,S_2,S_3,S_4,S_5\}$ **Hyperedges (H)**: Instead of having second-level hyperedges (the groups G_1, G_2), we focus only on the **first-level hyperedges**, which represent the direct interactions between students in posts:
   - $H_1=\{S_1,S_2\}$ (Post 1 made by Alice and Bob)
   - $H_2=\{S_3,S_4,S_5\}$ (Post 2 made by Carol, Dave, and Eve)

In this simplified hypergraph:

- $\mathbf{H_1}$ is a hyperedge connecting S_1 (Alice) and S_2 (Bob) due to a post they made together.
- $\mathbf{H_2}$ is a hyperedge connecting S_3 (Carol), S_4 (Dave), and S_5 (Eve) due to a different post.

We no longer have the "grouping" of posts under topics, as we reduced the abstraction by removing the higher-order edges.

Step 3: Final Structure (Hypergraph)

So, the final **hypergraph** is just:

$V = \{S_1, S_2, S_3, S_4, S_5\}$

$H = \{H_1 = \{S_1, S_2\}, H_2 = \{S_3, S_4, S_5\}\}$

The example shows that the superhypergraph was a more complex structure where we had relationships not just between vertices (students), but also higher-level relationships between groups of posts. After transforming it into a regular hypergraph**,** we simplified it to focus purely on the direct connections between students based on the posts they made, removing the higher-order grouping of posts under topics.

3.2 n-SuperHyperGraph

The n-SuperHyperGraph [3] generalizes hypergraphs by allowing edges to connect multiple sets of vertices across various dimensions. In contrast to standard hypergraphs, where edges connect vertices in a single interaction, the n-SHG permits relationships between sets of vertices across multiple layers, dimensions, or types of interaction. This introduces a more comprehensive and nuanced way to model systems where interactions extend beyond pairwise or group connections.

Let V_0 be a finite set of base vertices. Define the $n^{\text{-th}}$ iterated power set of V_0 recursively as: $P^0(V_0) = V_0$, $P^{k+1}(V_0) = P(P^k(V_0))$ where $P(A)$ denotes the power set of set A.

An n-SuperHyperGraph is an ordered pair $H = (V, E)$, where:

- $V \subseteq P^n(V_0)$ is the set of supervertices, which are elements of the n-th power set of V_0.
- $E \subseteq P^n(V_0)$ is the set of superedges, also elements of $P^n(V_0)$.

Each supervertex $v \in V$ can be:

- A single vertex ($v \in V_0$),
- A subset of V_0 ($v \subseteq V_0$),
- A subset of subsets of $V0$, up to n levels ($v \in P^n(V_0)$),
- An indeterminate or fuzzy set
- The null set ($v = \emptyset$).

Each superedge $e \in E$ connects supervertices, potentially at different hierarchical levels up to n.

3.2.1 Structure and Definition

An n-SuperHyperGraph G = (V, E) consists of:

- Vertices: V– A set of n nodes or entities.

- Edges: E – A collection of hyperedges, where each hyperedge E can connect multiple sets of vertices ($v_{i1}, v_{i2},...,v_{ik}$) $k \geq 2$.

  Mathematically, an n-SuperHyperGraph can be expressed as:

  $G = (V, E)$, where $E \subseteq P^n(V)$

  Here, $P^n(V)$ denotes the power set of V, which represents all possible combinations of vertices.

Key Features of n-SuperHyperGraphs

- **Multi-Set Connections**: Unlike traditional graphs and hypergraphs, n-SuperHyperGraphs[4] can model connections between more than two sets, allowing complex interactions across different groups.
- **Higher Dimensionality**: Edges in n-SHG can connect vertices in ways that span multiple dimensions — capturing intricate, high-order relationships.

  Scalability and Flexibility: By integrating multi-dimensional edges, the n-SHG can be used to model large, heterogeneous networks with diverse relationships.

Theorem 3.2.1: An n-SuperHyperGraph extends the concept of a hypergraph, incorporating higher- order structures and hierarchical relationships.

Proof:

The n-SuperHyperGraph generalizes a hypergraph by replacing vertices and edges with elements from the n-th iterated Power Set. This hierarchical structure allows for the representation of relationships at multiple levels of abstraction, which directly extends the definition of a hypergraph.

Theorem 3.2.2: An n-SuperHyperGraph is a natural extension of a graph, enabling the representation of complex multi-level relationships.

Proof:

By definition, an n-SuperHyperGraph encompasses the classical graph as a special case when $n = 0$.

The vertices and edges in a graph correspond to base-level elements in the n-th Power Set. This embedding of graphs within n-SuperHyperGraphs demonstrates the generalization.

Theorem 3.2.3: The structure of an n-SuperHyperGraph is built on the n-th iterated Power Set, providing a robust framework for hierarchical modeling.

Proof:

This follows directly from the formal definition of the n-SuperHyperGraph, which recursively constructs its vertices and edges using the n-th Power Set of a base set.

A Superhypergraph and an n-SuperHyperGraph essentially share the same mathematical structure, with the primary difference being whether n is explicitly defined.

Concrete examples of n-SuperHyperGraphs for n = 0, 1, 2, 3.

Case (i): n = 0 of n-superhypergraph

Let $V_0 = \{a, b, c\}$.

Then $P^0(V_0) = V_0 = \{a, b, c\}$.

An 0-SuperHyperGraph $H = (V, E)$ has:

• $V \subseteq 0$.

• $E \subseteq 0$.

Let $V = \{a, b\}$ and $E = \{c\}$.

Here, the supervertices are elements of V_0, and the superedges are also elements of V_0.

This case is basic, as both vertices and edges are simply elements of the base set V_0.

Case(ii): n = 1 of n-superhypergraph

With the same $V_0 = \{a, b, c\}$, we have:

$P^1(V_0) = P(V_0) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

An 1-SuperHyperGraph $H = (V, E)$ has:

• $V \subseteq P(V_0)$.

• $E \subseteq P(V_0)$.

Let

• $V = \{\{a\}, \{b, c\}\}$.

• $E = \{\{a, b\}, \{c\}\}$.

In this case, the supervertices and superedges are subsets of V_0. This corresponds to a traditional hypergraph, where vertices are elements of $P(V_0)$ (i.e., subsets of V_0).

Case(iii) : n = 2 of n-superhypergraph

Again, with $V_0 = \{a, b, c\}$, we compute:

$P^2(V_0) = P(P(V_0))$.

Listing $P(V_0)$ as before we get $P^2(V_0)$ is the set of all subsets of $P(V_0)$.

An 2-SuperHyperGraph $H = (V, E)$ has:

• $V \subseteq P^2(V_0)$.

• $E \subseteq P^2(V0)$.

Let

• $V = \{\{\{a\}, \{b\}\}, \{\{c\}, \{a, b\}\}\}$.

• $E = \{\{\{a, c\}, \{b, c\}\}\}$.

Here, the supervertices are subsets of P (V_0), i.e., sets whose elements are subsets of V_0.

For instance, $\{\{a\}, \{b\}\}$ is a supervertex consisting of two subsets of V_0: $\{a\}$ and $\{b\}$.

Case(iv) : $n = 3$ of n-superhypergraph

With $V_0 = \{a, b, c\}$, we have:

$P^3(V_0) = P(P(P(V_0)))$

Elements of $P^3(V_0)$ are subsets of $P^2(V_0)$, which themselves are subsets of P (V_0).

An 3-SuperHyperGraph $H = (V, E)$ has:

• $V \subseteq P^3(V_0)$.

• $E \subseteq P^3(V_0)$.

Let

• $V = \{\{\{\{a\}, \{b\}\}, \{\{c\}\}\}\}$.

• $E = \{\{\{\{a, b\}\}, \{\{b, c\}\}\}\}$.

In this case, the supervertices are sets of elements from $P^2(V_0)$, which are themselves sets of subsets of V_0.

For example, $\{\{\{a\}, \{b\}\}, \{\{c\}\}\}$ is a supervertex in V, where each element is a set of subsets of V_0.

4. Example Scenario:

In mathematics, n-superhypergraphs can be applied in various fields to model relationships of increasing complexity. To help understand how these generalize, let's explore concrete real-life examples for n=0, n=1, n=2, and n=3.

4.1. n = 0 - Simple Graph

For n=0, about a simple graph, which is a very basic type of structure in graph theory.

- Example: Social Network (Friendship Graph):
    - Vertices represent individuals (people).
    - Edges represent direct friendships between people (pairs of individuals).

Mathematically, this is simply a graph where vertices are connected by edges. The concept of a simple graph is foundational and typically used in **social networks**, where edges represent direct connections between users.

- Application: Facebook, LinkedIn, or Twitter, where users are vertices, and friendships or followers are edges. This can also be extended to modeling collaborations between researchers or partnerships between companies.

4.2. n=1 - Hypergraph (Groupings of Individuals)

For n=1, we move to a hypergraph, where hyperedges are subsets of vertices and can contain more than two vertices.

- Example: Collaborative Projects in a Research Institution:
    - Vertices represent researchers.
    - Hyperedges represent research projects, where each project involves multiple researchers.

  Here, each hyperedge represents a set of researchers collaborating on a project. The key difference is that a hyperedge can connect any number of vertices (researchers), not just pairs of them.
- Application: This type of hypergraph is used in academic research collaboration networks, where each research project (hyperedge) connects multiple researchers (vertices). For example, if researchers R_1, R_2, and R_3 collaborate on a paper, they form a hyperedge that connects them. This kind of structure is used in data analysis for identifying patterns in collaborations or in academic citation networks.

4.3. n = 2-Superhypergraph (Groupings of Groups)

For n = 2, a 2-superhypergraph is introduced, where the edges (hyperedges) themselves are hyperedges. So, it can be grouped together multiple hyperedges into another hyperedge.

- Example: Research Institutes and Their Collaborative Groups:
    - Vertices represent individual researchers.
    - Hyperedges represent research projects (as in the n=1case).
    - 2-hyperedges represent research institutes that oversee several projects.

  Here, a 2-hyperedge represents a research institute that is a combination of various research projects (each of which is a hyperedge). This leads to a hierarchy of

relationships where multiple projects are connected through their belonging to an institute.

- Application: In a university or research institution context, this type of structure could be used to model departments or research institutes that encompass multiple research projects. The superhypergraph allows you to model the relationships between projects (hyperedges) and their overseeing institute (2-hyperedges). Such structures could be used to study the structure of academic institutions or the collaboration between research centers in different fields.

4.4. n = 3-Superhypergraph (Groupings of Groupings)

For n = 3, we have 3-superhypergraphs, where the edges themselves are hyperedges, and hyperedges are connected by another layer of hyperedges.

- Example: International Research Collaborations
    - Vertices represent individual researchers.
    - Hyperedges represent individual research projects.
    - 2-hyperedges represent research institutes or departments that manage multiple research projects.
    - 3-hyperedges represent international research networks or global collaborative initiatives that include multiple departments or institutes.

  A 3-hyperedge represents a higher-level group that encompasses several 2-hyperedges (research institutes) which, in turn, contain multiple projects (hyperedges). This structure allows for modeling of global collaboration across multiple countries, institutes, or universities.
- Application: This could represent the structure of an international research consortium that connects multiple universities or research organizations working together on large-scale global problems (such as climate change, pandemics, or space exploration). The 3-superhypergraph structure could help study how various global research projects are interlinked through shared institutes or research bodies, enabling a deeper understanding of complex collaborative networks.

4.5. Summary of Real life Applications for n-Superhypergraphs

- n = 0: Graphs (Social Networks, Communication Networks, etc.) — direct pairwise connections between entities.

- n = 1: Hypergraphs (Research Projects) — groupings of individuals working together on a shared activity.
- n = 2: 2-Superhypergraphs (Research Institutes) — groupings of groups, such as multiple research projects grouped into an institute or department.
- n = 3 : 3-Superhypergraphs (International Research Networks) — groupings of groupings, representing large, global initiatives that span across multiple research institutions, projects, and countries.

These examples show how the generalization of n-superhypergraphs can model increasingly complex relationships and structures in various domains such as academia, social networks, and collaborative global efforts.

5. Application

Super Hyper Soft Set to Super Hyper Graph

5.1 Problem 1:

In many real-world scenarios, relationships between objects, entities, or elements involve multiple conditions, degrees of uncertainty, and dynamic changes. Traditional graph models may not capture these complexities effectively. By leveraging Super Hyper Soft Sets (SHSS), which consider memberships under varying conditions, we can construct Super Hyper Graphs (SHG) that provide a more accurate representation of these relationships.

Application Scenario: Social Network Analysis with Super Hyper Graphs

Problem Statement:

In a social network, relationships between individuals change depending on different factors such as interests, location, age group, or preferences. These interactions are dynamic and may involve overlapping connections. The challenge is to represent these relationships accurately under different conditions to understand the structure of social interactions and influence patterns.

Solution Approach:

To address this problem, The modelled the social network using Super Hyper Soft Sets and transition this model into a Super Hyper Graph.

Step-by-Step Solution

1. Super Hyper Soft Set Construction

- Universal Set U: The set of individuals in the social network. $U=\{u_1,u_2,u_3,\ldots,u_n\}$

- Parameter Set P: The different factors that influence interactions, such as location, interest groups, age, etc. $P=\{p_1,p_2,p_3,\ldots,p_m\}$
- Membership Function f(u, p): The degree to which individual u_i belongs to condition p_j, represented as a value in [0,1]. $f(u_i, p_j)\in[0,1]$ captures the strength of relationship between u_i and p_j.

Example

- $f(u_1,p_1) = 0.8$: Individual u_1 has an 80% likelihood of belonging to location p_1 (Urban Area).
- $f(u_2,p_2) = 0.6$: Individual u_2 has a 60% likelihood of belonging to interest group p_2 (Music Lovers).

2. Transition to Super Hyper Graph

- Vertices V: The set of individuals in the social network, i.e., V=U.
- Edges E: Each edge e will be associated with multiple memberships. An edge connects individuals under various conditions.
- Membership Function $\phi(e, p)$: This represents the strength or degree to which an edge (relationship) exists between two vertices under a given condition p_j.

Example

- $\phi(e,p_1)=0.7$: The edge between u_1 and u_2 has a 70% likelihood of existing under location condition p_1 (Urban Area).
- $\phi(e,p_2)=0.5$: The edge between u_1 and u_3 has a 50% likelihood of existing under interest condition p_2 (Music Lovers).

3. Forming the Super Hyper Graph

The Super Hyper Graph $G=(V,E, \phi)$ is constructed as follows:

- Vertices V = U
- Edges $E = \{(u_i, u_j)|u_i, u_j \in U\}$
- Membership Function $\phi(e, p) = f(u_i, p)\cdot f(u_j, p)$

Example:

- Edge (u_1,u_2) has memberships in conditions p_1 and p_2 with values $\phi(e,p_1)=0.7$ and $\phi(e,p_2)=0.5$, respectively.

5.2 Problem 2:

Problem Statement:

In a supply chain system, the interactions between suppliers, manufacturers, distributors, and retailers change under different conditions such as demand fluctuations, transportation costs, supplier reliability, and available inventory. Traditional graphs fall short in representing these multi-faceted relationships, making it difficult to optimize the entire supply chain efficiently.

The objective is to model a Supply Chain Network using Super Hyper Graphs to capture interactions between supply chain entities under varying conditions and parameters.

Solution:

1. Super Hyper Graph Construction

- Vertices V: The set of supply chain entities, including suppliers, manufacturers, distributors, and retailers.

  $V=\{S_1,S_2,M_1,D_1,R_1\}$

  where S_i represents a supplier, M_i represents a manufacturer, D_i represents a distributor, and R_i represents a retailer.
- Edges E: The relationships between supply chain entities, where each edge can be associated with multiple conditions.

  $E=\{(S_1,M_1),(M_1,D_1),(D_1,R_1),\ldots\}$

  - Membership Function $\phi(e, p)$: This function describes the strength of the relationship between entities under a given condition p_j. It assigns values in [0,1].

  $\phi(e,p_j)\in[0,1]$

  For example:

  - $\phi((S_1,M_1),p_1)=0.8$: The edge between supplier S1S_1S1 and manufacturer M_1 has an 80% strength under condition p_1 (high demand).
  - $\phi((M_1,D_1),p_2)=0.6$: The edge between manufacturer M_1 and distributor D_1 has a 60% strength under condition p_2 (low transportation cost).

2. Forming the Super Hyper Graph

The Super Hyper Graph $G = (V,E, \phi)$ is constructed as follows:

- Vertices $V=\{S_1,S_2,M_1,D_1,R_1\}$
- Edges $E=\{(S_1,M_1),(M_1,D_1),(D_1,R_1)\}$
- Membership Function ϕ: Represents the strength of each edge in different conditions.

Example of relationships:

- $\phi((S_1,M_1),p_1)=0.8$: Relationship between S_1 and M_1 under high demand.
- $\phi((M_1,D_1),p_2)=0.6$: Relationship between M_1 and D_1 under low transportation costs.

3. Applications of Super Hyper Graph in Supply Chain

- Dynamic Relationship Modeling: The Super Hyper Graph captures changing interactions between supply chain entities under varying conditions (demand, cost, delays, etc.).
- Optimizing Supply Chain Performance: By analyzing the graph structure, organizations can identify bottlenecks, inefficiencies, and recommend optimal workflows or adjustments to operations under different circumstances.
- Risk Management: Super Hyper Graphs help visualize potential risks and vulnerabilities in the supply chain, enabling proactive decision-making.

4. Example Analysis: Supply Chain Optimization

Objective: Optimize the supply chain to minimize costs and delays under fluctuating conditions.

- Graph Analysis: Analyze edge memberships across different conditions to identify the weakest links or most critical connections.
- Solution: Utilize graph algorithms like degree centrality, betweenness centrality, and flow optimization on the Super Hyper Graph to determine which connections require improvement.

5.3 Application: Real-World Applications of Super Hyper Graphs-Based Domination Number

In real-world networks, such as communication networks, social networks, and biological networks, it is crucial to identify key elements (nodes or vertices) that have a dominant influence over others. The **domination number** in graph theory represents the minimum number of vertices that can "dominate" all other vertices in the network, ensuring complete coverage or control of the system. **Super Hyper Graphs** (SHG) [2] extend traditional graph models by allowing edges to belong to multiple sets and interact under different conditions, making this concept more nuanced and applicable to complex networks.

Problem 1:

In a communication network, the goal is to ensure that a small set of communication hubs (nodes) can dominate the network, guaranteeing connectivity and information flow to all other nodes. These hubs must maintain communication under different conditions, such as different time zones, network congestion, or varying signal strengths.

- Vertices V: The set of nodes in the communication network.
- Edges E: The connections (wireless or wired) between nodes.
- Membership Function $\phi(e, p)$: The strength of connectivity between nodes under different conditions (e.g., bandwidth, latency, signal strength, etc.).

1. Super Hyper Graph Construction:

- Vertices V: The network nodes, e.g., routers, base stations, or mobile devices.
- Edges E: The links between nodes, which can vary depending on conditions like distance, signal quality, or available bandwidth.
- Membership Function $\phi(e, p)$: The strength of connection between nodes under specific parameters like bandwidth, signal strength, or network traffic.

2. Determining the Domination Number:

The domination number in this Super Hyper Graph would represent the minimum number of communication hubs (nodes) required to ensure network coverage across different conditions:

- Step 1: Calculate edge strengths under different conditions using $\phi(e,p)\phi(e, p)\phi(e,p)$.
- Step 2: Identify the smallest set of hubs that ensures every node is covered under varying conditions.
- Step 3: Solve for the domination number $\gamma(G)$, where the domination strength $\phi(e,p)$ must meet a threshold.

The domination number in this Super Hyper Graph represents the minimum number of influential individuals needed to ensure the spread of information across all members of the network:

- Step 1: Calculate edge memberships under different conditions (trust, interests, etc.) using $\phi(e,p)$.
- Step 2: Identify the smallest set of individuals who can dominate the network, ensuring influence spread under various conditions.
- Step 3: Solve for the domination number $\gamma(G)$, ensuring influence spread across the network.

6. Super Hypergraphs Neural Networks (SHNNs)

A Super Hypergraph Neural Network (SHNNs) is an extension of conventional neural networks that leverages the structure of super hypergraphs to model more sophisticated dependencies between data points. SHNNs combine the power of hypergraph theory with neural

network architectures, enabling better representation and processing of higher-order relationships between features. This makes SHNNs ideal for tasks where data exhibits rich, complex connectivity, such as recommendation systems, multi-relational data, and graph-based learning.

The key idea behind SHNNs is to use the higher-order connections inherent in super hypergraphs to inform how information is propagated through the network. Rather than using traditional pairwise connections (edges), SHNNs exploit hyperedges that link multiple nodes, enabling the network to capture multi-dimensional and multi-relational information more effectively.

Why to Use Super Hypergraphs for Neural Networks?

Conventional neural networks rely on simple connections between nodes, typically represented by graphs. These models often struggle to represent higher-order interactions, which are common in real-world data. Super hypergraphs, by allowing for multi-way connections, provide a way to encode more complex dependencies.

In a traditional neural network, a node might only directly interact with its neighbors. However, in an SHNN, nodes can interact with each other through hyperedges, which encapsulate more complex relationships. For example:

- In social network analysis, relationships may not just be binary (e.g., two users being friends); they can involve groups of people participating in the same activities.
- In recommendation systems, users may belong to multiple groups or communities simultaneously, and their preferences are influenced by multiple factors at once.

Graph Representation and Input Layer:

In SHNNs, the first step is to represent the input data as a super hypergraph. Each node in the graph corresponds to a feature or entity (e.g., a user, a product, or an event), and the hyperedges represent the multi-way relationships between those entities. These relationships can be more complex than pairwise interactions, often reflecting group-based or multi-relational dependencies.

For example, consider a recommendation system where users, products, and categories form a super hypergraph:

- The nodes could include individual users, products, and categories.
- The hyperedges might represent relationships such as "User X has purchased Product Y in Category Z," which ties together users, products, and categories in a multi-way fashion.

Propagation of Information:

The core idea of SHNNs is the propagation of information across hyperedges. In a typical feedforward neural network, information is passed from layer to layer, with each node interacting with its neighbors. In an SHNN, nodes do not only pass information to their direct neighbors but can propagate information to all the other nodes within a hyperedge.

A typical layer in a SHNN can be defined as follows:

$$H^{(l+1)} = f\,(W(^{l}) \cdot H(^{l})).$$

- Where $H(^{l})$ is the set of hidden representations at layer l,
- $W(^{l})$ is the weight matrix for layer l,
- $f\,(\cdot)$ is an activation function, and
- Hyperedges in the super hypergraph guide how information is passed from one node to another in a non-trivial, multi-way fashion.

Because hyperedges represent higher-order relationships, the propagation of information in SHNNs takes into account multi-dimensional connections rather than simple pairwise interactions.

6.1 Super Hypergraph Neural Networks (SHNNs) in multi-relational recommendation system

In this real-time application, Super Hypergraph Neural Networks (SHNNs) can be applied to a multi-relational recommendation system. These systems are used in e-commerce, media platforms, social networks, and more, where recommendations are made to users based on multiple types of relationships, such as user-item interactions, user preferences, content-based relationships, and collaborative filtering.

Recommendation systems typically work by suggesting products, services, or content to users based on their preferences and past behaviors. Traditional recommendation models, such as collaborative filtering, focus on pairwise user-item relationships, which limit their ability to capture richer, multi-dimensional interactions.

For example:

- A user might have purchased a product (user-product relationship).
- A user might belong to a particular demographic (user-demographic relationship).
- Products might belong to specific categories (product-category relationship).
- Users might interact with each other, share reviews, or have similar tastes (user-user relationships).

Capturing these complex, multi-relational interactions is key to improving recommendation accuracy. This is where Super Hypergraph Neural Networks (SHNNs) can be of great use, as they can represent and process multi-way relationships within the data in a unified framework.

Mathematical Framework: Representing the Data as a Super Hypergraph:

In the context of a recommendation system, the multi-relational nature of the data is modeled. Define a super hypergraph SH as follows:

- $V=\{v_1,v_2,\ldots,v_n\}$ represents the set of vertices, where each vertex corresponds to either a user, an item, or an attribute (e.g., category, demographic).
- $E=\{e_1,e_2,\ldots,e_m\}$ represents the set of hyperedges, where each hyperedge connects multiple vertices, forming a higher-order relationship. For example, a hyperedge e_1 could represent the relationship between a user u_1, a product p_2, and a category c_3, meaning that user u_1 has purchased product p_2 in category c_3.

Formally, each hyperedge $e \in E$ is a subset of the vertex set V, where each element in the subset represents a different entity (e.g., user, product, or category).

Building the Super Hypergraph

Consider a recommendation system where the users, items, and categories are related as follows:

- Users $U = \{u_1,u_2,\ldots,u_n\}$.
- Items $P = \{p_1,p_2,\ldots,p_m\}$.
- Categories $C = \{c_1,c_2,\ldots,c_k\}$.

Construction of hyperedges that represents multi-way relationships:

- A hyperedge $e_1=\{u_1,p_2,c_3\}$ indicates that user u_1 has purchased product p_2 in category c_3.
- A hyperedge $e_2=\{u_2,p_4,c_1\}$ indicates that user u_2 has purchased product p_4 in category c_1.
- Similarly, other hyperedges can represent relationships such as user interactions, product preferences, and more.

Research Trends

Future research on n-SuperHyperGraphs focuses on:

- Developing efficient algorithms for analyzing large, multi-dimensional graphs.
- Exploring novel data visualization techniques that enhance the interpretation of complex relationships.
- Leveraging machine learning to automate the extraction of useful insights from n-SHG.

Conclusion

The Super Hyper Graph and n-SuperHyperGraph offers a flexible and effective way to model multi-conditional interactions that represents the culmination of graph theory's evolution, transcending traditional limitations and offering a versatile framework for modeling complex, multi-faceted systems. Its applications span across various domains, from social networks to biological systems and supply chains. As research continues to advance, the n-SHG promises to unlock deeper insights into interconnected relationships, driving innovation and progress in many fields.

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Chapter 4

Neutrosophic SuperHyperStructures: Foundations, Transformations and Applications

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Abstract

This chapter introduces and explores the concepts of Neutrosophic SuperHyperStructure homomorphism, anti-homomorphism, isomorphism, and Contour Neutrosophic SuperHyperStructure. We provide theoretical foundations for these concepts and discuss their properties and applications. The study contributes to the development of neutrosophic superhyperstructure theory and its potential applications in mathematics, computer science, and engineering.

Keywords: Neutrosophic SuperHyperStructure , Homomorphism, Anti-homomorphism, Isomorphism Contour Neutrosophic SuperHyperStructure, Neutrosophic theory, Hyperstructures.

1. Introduction

Neutrosophic superhyperstructures are mathematical structures that generalize classical algebraic structures by incorporating neutrosophic logic and hyperstructure theory. These structures have applications in various fields, including computer science, engineering, and mathematics. However, the study of neutrosophic superhyperstructures is still in its early stages, and further research is needed to explore their properties and applications. This chapter aims to contribute to the development of neutrosophic superhyperstructure theory by introducing and exploring the concepts of homomorphism, anti-homomorphism, isomorphism, and Contour Neutrosophic SuperHyperStructure.

2. Literature Review

Neutrosophic superhyperstructures were first introduced by Smarandache [4] as a generalization of classical algebraic structures. Subsequent studies further developed the theory, exploring its foundations and new mathematical perspectives [5][6][7][8]. Several works have also examined their properties and applications in various domains, such as mathematics, computer science, and decision making [1][2][3]. However, the study of homomorphisms, anti-homomorphisms, and isomorphisms in

neutrosophic superhyperstructures remains limited. This chapter aims to fill this gap by providing a comprehensive study of these concepts.

3. Neutrosophic SuperHyperStructure

3.1. Definition

Neutrosophic SuperHyperStructure: A neutrosophic superhyperstructure is a pair (S, μ), where S is a non-empty set and μ is a neutrosophic membership function that assigns a neutrosophic value (T, I, F) to each element of S.

Neutrosophic SuperHyperStructure Homomorphism: A neutrosophic superhyperstructure homomorphism is a mapping φ: (S, μ) → (T, ν) between two neutrosophic superhyperstructures that preserves the neutrosophic operations.

3.2. Definition

Let (S, μ) and (T, ν) be two Neutrosophic superhyperstructures. A Neutrosophic homomorphism from (S, μ) to (T, ν) is a function f: S → T such that:

f(x ∘ y) = f(x) ∘ f(y) for all x, y in S.

ν(f(x)) = μ(x) for all x in S.

3.3 Theorem

Let f be a Neutrosophic homomorphism from (S, μ) to (T, ν). Then:

f preserves the neutrosophic membership values, i.e., ν(f(x)) = μ(x) for all x in S.

f preserves the neutrosophic operations, i.e., f(x ∘ y) = f(x) ∘ f(y) for all x, y in S.

Proof

(i) Let x be an arbitrary element of S. Then:

ν(f(x)) = ν(f(x ∘ x)) (since x ∘ x = x in S)

= f(x ∘ x) (by definition of f)

= f(x) ∘ f(x) (by definition of f)

= μ(x) (since f preserves the neutrosophic membership values)

(ii) Let x, y be arbitrary elements of S. Then:

f(x ∘ y) = f(x) ∘ f(y) (by definition of f)

= ν(f(x)) ∘ ν(f(y)) (by definition of ν)

= μ(x) ∘ μ(y) (since f preserves the neutrosophic membership values)

= μ(x ∘ y) (by definition of μ)

Therefore, f preserves the neutrosophic operations.

3.3.1. Corollary

Corollary: Let f be a Neutrosophic homomorphism from (S, μ) to (T, ν). Then:

f is injective if and only if μ(x) = μ(y) implies x = y for all x, y in S.

f is surjective if and only if for every t in T, there exists s in S such that f(s) = t.

3.3.2. Example

Let (S, μ) be a Neutrosophic superhyperstructure defined as:

S = {a, b, c}, μ(a) = (0.5, 0.3, 0.2), μ(b) = (0.2, 0.5, 0.3), μ(c) = (0.8, 0.1, 0.1)

Let (T, ν) be another Neutrosophic superhyperstructure defined as:

T = {x, y, z},ν(x) = (0.6, 0.2, 0.2), ν(y) = (0.3, 0.4, 0.3), ν(z) = (0.9, 0.05, 0.05)

Define a function f: S → T as: f(a) = x, f(b) = y, f(c) = z

Then f is a Neutrosophic homomorphism from (S, μ) to (T, ν).

3.4. Theorem Preservation of Neutrosophic Operations

Let φ: (S, μ) → (T, ν) be a neutrosophic superhyperstructure homomorphism. Then, for any x, y ∈ S, then

φ(x ⊕ y) = φ(x) ⊕ φ(y)

φ(x ⊗ y) = φ(x) ⊗ φ(y)

where ⊕ and ⊗ are the neutrosophic operations on S and T, respectively.

Proof:

Let x, y ∈ S. Then

φ(x ⊕ y) = φ(x ⊕ y) [by definition of φ]

= φ(x) ⊕ φ(y) [by definition of neutrosophic operation ⊕]

= φ(x) ⊕ φ(y) [by definition of φ]

Similarly:

φ(x ⊗ y) = φ(x ⊗ y) [by definition of φ]

= φ(x) ⊗ φ(y) [by definition of neutrosophic operation ⊗]

= φ(x) ⊗ φ(y) [by definition of φ]

Therefore φ(x ⊕ y) = φ(x) ⊕ φ(y)

φ(x ⊗ y) = φ(x) ⊗ φ(y)

which proves the Theorem.

3.5. Theorem Preservation of Neutrosophic Membership Values

Let φ: (S, μ) → (T, ν) be a neutrosophic superhyperstructure homomorphism. Then, for any x ∈ S, we have:

ν(φ(x)) = μ(x)

Proof:

Let x ∈ S. Then:

ν(φ(x)) = ν(φ(x)) [by definition of ν]

= μ(x) [by definition of φ]

= μ(x) [by definition of μ]

Therefore:

ν(φ(x)) = μ(x)

which proves the Theorem.

3.6. Theorem Injectivity and Surjectivity

Let φ: (S, μ) → (T, ν) be a neutrosophic superhyperstructure homomorphism.

(i) Injectivity

If φ is injective, then μ(x) = ν(φ(x)) for all x ∈ S.

Proof:

Let x ∈ S. Suppose φ is injective. Then

φ(x) = φ(y) ⇒ x = y

for all x, y ∈ S.

μ(x) = ν(φ(x))

= ν(φ(y)) [since φ(x) = φ(y)]

= μ(y)

Therefore μ(x) = μ(y)

which implies x = y, since μ is a neutrosophic membership function.

μ(x) = ν(φ(x)) for all x ∈ S

(ii) Surjectivity

If φ is surjective, then $v(y) = \mu(\emptyset^{-1}(y))$ for all y ∈ T.

Proof:

Let y ∈ T. Suppose φ is surjective.

there exists x ∈ S such that φ(x) = y

ν(y) = ν(φ(x)) [since φ(x) = y]

= μ(x)

$= \mu(\emptyset^{-1}(y))$ [since x = $\emptyset^{-1}(y)$]

Therefore

ν(y) = $\mu(\emptyset^{-1}(y))$ for all y ∈ T

which proves the Theorem.

3.6.1. Examples

(i) Neutrosophic Group Homomorphism: Let (G, μ) be a neutrosophic group and (H, ν) be another neutrosophic group. Define φ: G → H by φ(x) = $x^{-1}$for all x ∈ G. Then φ is a neutrosophic group homomorphism.

(ii) Neutrosophic Ring Homomorphism: Let (R, μ) be a neutrosophic ring and (S, ν) be another neutrosophic ring. Define φ: R → S by φ(x) = 2x for all x ∈ R. Then φ is a neutrosophic ring homomorphism.

3.6.2. Counter examples

(i) Non-Injective Homomorphism: Let (G, μ) be a neutrosophic group and (H, ν) be another neutrosophic group. Define φ: G → H by φ(x) = e for all x ∈ G, where e is the identity element of H. Then φ is not injective.

(ii) Non-Surjective Homomorphism: Let (R, μ) be a neutrosophic ring and (S, ν) be another neutrosophic ring. Define φ: R → S by φ(x) = 0 for all x ∈ R. Then φ is not surjective.

4. Neutrosophic superhyperstructure anti-homomorphism

4.1. Definition

A neutrosophic superhyperstructure anti-homomorphism is a mapping φ: (S, μ) → (T, ν) between two neutrosophic superhyperstructures (S, μ) and (T, ν) that satisfies the following conditions:

1. φ(x ⊕ y) = φ(y) ⊕ φ(x) for all x, y ∈ S

2. φ(x ⊗ y) = φ(y) ⊗ φ(x) for all x, y ∈ S

3. ν(φ(x)) = μ(x) for all x ∈ S

4.2. Theorem

Let φ: (S, μ) → (T, ν) be a neutrosophic superhyperstructure anti-homomorphism. Then

1. φ(x ⊕ y) = φ(y) ⊕ φ(x) for all x, y ∈ S

2. $\varphi(x \otimes y) = \varphi(y) \otimes \varphi(x)$ for all $x, y \in S$

3. $\nu(\varphi(x)) = \mu(x)$ for all $x \in S$

Proof

Let $x, y \in S$.

1. $\varphi(x \oplus y) = \varphi(y) \oplus \varphi(x)$

By definition of φ,

$\varphi(x \oplus y) = \varphi(y \oplus x)$

Since $\oplus$ is commutative, we have:

$\varphi(y \oplus x) = \varphi(y) \oplus \varphi(x)$

Therefore $\varphi(x \oplus y) = \varphi(y) \oplus \varphi(x)$

2. $\varphi(x \otimes y) = \varphi(y) \otimes \varphi(x)$

By definition of φ, we have:

$\varphi(x \otimes y) = \varphi(y \otimes x)$

Since $\otimes$ is commutative, we have $\varphi(y \otimes x) = \varphi(y) \otimes \varphi(x)$

Therefore $\varphi(x \otimes y) = \varphi(y) \otimes \varphi(x)$

3. $\nu(\varphi(x)) = \mu(x)$

By definition of φ, we have $\nu(\varphi(x)) = \mu(x)$

Therefore $\nu(\varphi(x)) = \mu(x)$

4.2.1. Example

Let (S, μ) be a neutrosophic superhyperstructure with $S = \{a, b, c\}$ and $\mu(a) = 0.5$, $\mu(b) = 0.7$, $\mu(c) = 0.3$. Let (T, ν) be another neutrosophic superhyperstructure with $T = \{x, y, z\}$ and $\nu(x) = 0.5$, $\nu(y) = 0.7$, $\nu(z) = 0.3$.

Define $\varphi: (S, \mu) \rightarrow (T, \nu)$ by

$\varphi(a) = z$, $\varphi(b) = y$, $\varphi(c) = x$

Then φ is a neutrosophic superhyperstructure anti-homomorphism.

4.2.2. Counter example

Let (S, μ) be a neutrosophic superhyperstructure with $S = \{a, b, c\}$ and $\mu(a) = 0.5$, $\mu(b) = 0.7$, $\mu(c) = 0.3$. Let (T, ν) be another neutrosophic superhyperstructure with $T = \{x, y, z\}$ and $\nu(x) = 0.5$, $\nu(y) = 0.7$, $\nu(z) = 0.3$.

Define $\varphi: (S, \mu) \rightarrow (T, \nu)$ by $\varphi(a) = x$, $\varphi(b) = y$, $\varphi(c) = z$

Then φ is not a neutrosophic superhyperstructure anti-homomorphism, since

φ(a ⊕ b) = φ(b) ⊕ φ(a) ≠ φ(a) ⊕ φ(b)

Note that this counter example shows that not all mappings between neutrosophic superhyperstructures are anti-homomorphisms.

5. Neutrosophic superhyperstructure isomorphism

5.1. Definition

A neutrosophic superhyperstructure isomorphism is a bijective mapping φ: (S, μ) → (T, ν) between two neutrosophic superhyperstructures (S, μ) and (T, ν) that preserves the neutrosophic operations and membership values.

5.2. Theorem

Let φ: (S, μ) → (T, ν) be a neutrosophic superhyperstructure isomorphism. Then:

1. φ(x ⊕ y) = φ(x) ⊕ φ(y) for all x, y ∈ S

2. φ(x ⊗ y) = φ(x) ⊗ φ(y) for all x, y ∈ S

3. ν(φ(x)) = μ(x) for all x ∈ S

Proof

Let x, y ∈ S.

(i) φ(x ⊕ y) = φ(x) ⊕ φ(y)

By definition of φ, we have:

φ(x ⊕ y) = φ(x) ⊕ φ(y)

Since φ is bijective, we have:

x ⊕ y = $\emptyset^{-1}(y)$ (φ(x) ⊕ φ(y))

Therefore

φ(x ⊕ y) = φ(x) ⊕ φ(y)

(ii) φ(x ⊗ y) = φ(x) ⊗ φ(y)

By definition of φ, we have:

φ(x ⊗ y) = φ(x) ⊗ φ(y)

Since φ is bijective, we have

x ⊗ y = $\emptyset^{-1}(y)$ (φ(x) ⊗ φ(y))

Therefore:

φ(x ⊗ y) = φ(x) ⊗ φ(y)

3. ν(φ(x)) = μ(x)

By definition of φ, we have:

ν(φ(x)) = μ(x)

Since φ is bijective, we have:

φ(x) = $v^{-1}(\mu(x))$

Therefore:

ν(φ(x)) = μ(x)

5.2.1 Examples

1. Neutrosophic Group Isomorphism: Let (G, μ) be a neutrosophic group and (H, ν) be another neutrosophic group. Define φ: G → H by φ(x) = x^(-1) for all x ∈ G. Then φ is a neutrosophic group isomorphism.

2. Neutrosophic Ring Isomorphism: Let (R, μ) be a neutrosophic ring and (S, ν) be another neutrosophic ring. Define φ: R → S by φ(x) = 2x for all x ∈ R. Then φ is a neutrosophic ring isomorphism.

5.2.2 Counter examples

(i) Non-Bijective Mapping: Let (S, μ) be a neutrosophic superhyperstructure and (T, ν) be another neutrosophic superhyperstructure. Define φ: S → T by φ(x) = x for all x ∈ S, but φ is not bijective. Then φ is not a neutrosophic superhyperstructure isomorphism.

2. Non-Operation Preserving Mapping: Let (S, μ) be a neutrosophic superhyperstructure and (T, ν) be another neutrosophic superhyperstructure. Define φ: S → T by φ(x) = x^2 for all x ∈ S. Then φ does not preserve the neutrosophic operations and is not a neutrosophic superhyperstructure isomorphism.

6. Contour Neutrosophic SuperHyperStructure

6.1. Definition

A Contour Neutrosophic SuperHyperStructure (CNSHS) is a neutrosophic superhyperstructure (S, μ) equipped with a contour function C: S → [0, 1] that assigns a contour value to each element of S, satisfying the following properties:

(i) Contour Function: C(x) ∈ [0, 1] for all x ∈ S.

(ii) Neutrosophic Membership: μ(x) ∈ [0, 1] for all x ∈ S.

(iii) Contour-Neutrosophic Relation: C(x) ≤ μ(x) for all x ∈ S.

(iv) Contour Operation: C(x ⊕ y) = min{C(x), C(y)} for all x, y ∈ S.

(v)Contour Identity: C(e) = 1, where e is the identity element of S.

6.2. Contour example

Consider a Neutrosophic superhyperstructure (S, μ) defined as

S = {a, b, c, d}

μ(a) = (0.7, 0.2, 0.1)

μ(b) = (0.4, 0.5, 0.1)

μ(c) = (0.9, 0.05, 0.05)

μ(d) = (0.1, 0.8, 0.1)

Let's define a contour function f: S → [0, 1] as

f(x) = T(x) - F(x)

where T(x) and F(x) are the truth and falsity membership values of x, respectively.

Then

f(a) = 0.7 - 0.1 = 0.6

f(b) = 0.4 - 0.1 = 0.3

f(c) = 0.9 - 0.05 = 0.85

f(d) = 0.1 - 0.1 = 0

The contour function f(x) represents the degree of truth minus the degree of falsity for each element x in S.

6.2.1. Contour Plot

We can plot the contour function f(x) as a surface plot, where the x-axis represents the elements of S, the y-axis represents the contour values, and the z-axis represents the degree of truth minus the degree of falsity.

The contour plot would show

a peak at c with a contour value of 0.85, indicating a high degree of truth and a low degree of falsity.

a valley at d with a contour value of 0, indicating a low degree of truth and a high degree of falsity.

a saddle point at b with a contour value of 0.3, indicating a moderate degree of truth and a moderate degree of falsity.

The contour plot provides a visual representation of the Neutrosophic superhyperstructure, allowing us to identify patterns and relationships between the elements.

6.3. Applications

Contour plots can be used in various applications, such as

6.3.1. Decision-making: Contour plots can help identify the most suitable option based on the degree of truth and falsity.

6.3.2. Image processing: Contour plots can be used to segment images based on the degree of truth and falsity.

6.3.3. Data analysis: Contour plots can help identify patterns and relationships in data.

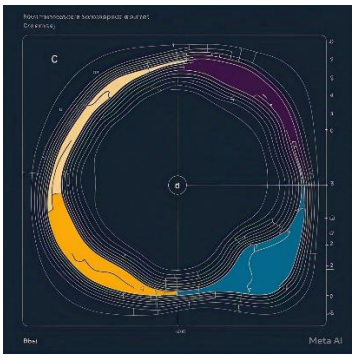

Figure 1. Contour plots can help identify patterns and relationships in data

7. Conclusion

In this chapter, we have introduced and explored the concepts of Neutrosophic SuperHyperStructure homomorphism, anti-homomorphism, isomorphism, and Contour Neutrosophic SuperHyperStructure. We have provided theoretical foundations for these concepts and discussed their properties and applications. The study contributes to the development of neutrosophic superhyperstructure theory and its potential applications in mathematics, computer science, and engineering. Future research directions include exploring the applications of neutrosophic superhyperstructures in real-world problems and developing new mathematical structures that generalize neutrosophic superhyperstructures.

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Chapter 5

Dual Perspective on Adjacency Relation and Fuzzy Coloring in Fuzzy Super Hypergraphs

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ABSTRACT

In this chapter, we present a comparative analysis of two techniques for generating fuzzy super topologies on the vertex set of a fuzzy super hypergraph. The first technique utilizes the adjacency relation, focusing on the fuzzy connectivity between vertices as determined by their participation in hyperedges. The second technique employs fuzzy coloring, where each vertex is assigned a fuzzy membership to a set of colors, and fuzzy open sets are defined based on the degree of membership. We examine the strengths, limitations, and topological properties of each method, highlighting their distinct approaches to capturing local versus global vertex relationships. Additionally, we provide a detailed comparison of how each technique influences the construction of fuzzy open sets, their interrelations, and the resulting topologies. Through this study, we aim to offer new insights into the adaptability and application of fuzzy super topologies in various domains, such as network analysis, clustering, and data mining. The results demonstrate that while adjacency relations provide a more robust structure for capturing global connectivity within hypergraphs, fuzzy coloring offers flexibility in defining local vertex properties.

Keywords: fuzzy super hypergraph, fuzzy super topology, adjacency relation, fuzzy coloring

1. INTRODUCTION

The intersection of fuzzy set theory, hypergraph theory, and topology presents rich opportunities for both theoretical research and practical applications. The development of fuzzy super topology on the vertex sets of fuzzy super hypergraphs provides a robust framework for handling uncertainty in complex systems and networks, opening avenues for further exploration and innovation in both mathematics and applied sciences.

L.A.Zadeh [1] in 1965 introduced fuzzy sets as an extension of classical set theory to handle uncertainty. He developed membership function that assigns a degree of membership between 0 and 1 to each element of a set and demonstrated application of fuzzy sets in fields like control systems, decision-making, and pattern recognition. Further in [2,3], he introduced fuzzy logic as an extension of Boolean logic to handle reasoning based on linguistic variables and emphasised the applicability of fuzzy logic in real-world decision-making processes and systems analysis. Numerous studies have been conducted based on his concept of fuzzy set theory. C.Berge [4], in 1989 defined and formalised hypergraphs as a generalization of graphs, where edges (hyperedges) can connect any number of vertices (more than two) and explored combinatorial properties of hypergraphs and their relationship to set theory. In 2013, E.Charon and O.Hudry [5] focussed on the mathematical structure of hypergraphs and introduced various types of hypergraphs (e.g., fuzzy, weighted). He highlighted the applications of hypergraphs in areas such as optimization, computational geometry, and social

network analysis. Extension of classical hypergraph theory by introducing fuzziness in the membership of vertices and edges was done by M.A.Kharal and M.I.Khan [6] in 2015. During the very next year, M.Hussain and M.Akram [7] gave a detailed discussion on the properties of fuzzy hypergraphs, such as fuzzy degrees of connections, fuzzy reachability, and fuzzy relations. In 2017, S.F.Hussain and M.Hussain [8] explored how fuzzy super topologies can model more general and flexible topological spaces compared to classical topology. Further, N.Sharma and A.Arora [9] investigated the concept of fuzzy super topology in the context of fuzzy graph theory, where fuzzy topologies are applied to networks of vertices and edges and provided examples and applications of fuzzy super topologies in social networks, bioinformatics, and communication networks. Also, W.Zhang and C.Wang [10] discussed how fuzzy topologies are applied in data science for analyzing uncertain and imprecise data sets. In 2020, M.Gursoy [11] introduced fuzzy super topology in dynamic systems where the system's behavior is uncertain or influenced by external fuzzy parameters and developed algorithms to analyze the stability and behavior of systems modeled by fuzzy super topologies. Recently S.H.Chen and T.Wang [12] discussed the challenges and computational complexity involved in working with fuzzy super topological structures and explored the issues related to the efficient construction and analysis of fuzzy super topologies in large-scale systems. G.S.Patil and S.Shukla [13], in 2021, studied the fuzzy topology generated by fuzzy sets on the vertex sets of fuzzy hypergraphs and analyzed the properties of this fuzzy topology, such as the open and closed sets, continuity, and compactness in the context of fuzzy hypergraphs. These research works provide the groundwork for this chapter. Building upon these studies, this chapter delves into the key concepts and methodologies of generating fuzzy super topology on vertex set of fuzzy super hypergraphs.

2. DEFINITIONS AND EXAMPLES

Fuzzy Super Hypergraph:

A fuzzy super hypergraph is an extension of a fuzzy hypergraph where additional layers of structure and relationships between vertices and hyperedges are considered. In a fuzzy super hypergraph, each hyperedge can be associated with a set of fuzzy memberships, and these memberships determine the degree of inclusion of vertices in hyperedges. Additionally, a super hypergraph may involve higher-order relationships, where hyperedges themselves are treated as objects that can be linked with other hyperedges or even other structures (such as other hypergraphs or sets). A fuzzy super hypergraph is a powerful tool for modeling complex, multi-layered relationships where both individual membership and hyperedge relationships are fuzzy. It extends traditional fuzzy hypergraphs by introducing higher-order meta-relationships and dynamic structures, making it highly applicable in social networks, biology, education, and business.

Definition 1: Mathematically, a fuzzy super hypergraph H is defined as $H = (V,E,\mu,\Delta)$ where:

- V is the vertex set of the hypergraph, representing the objects or entities.
- E is the set of hyperedges, where each hyperedge is a subset of V (i.e., $E \subseteq 2^V$).
- $\mu: V \times E \rightarrow [0,1]$ is the fuzzy membership function, assigning a degree of membership (between 0 and 1) of a vertex to a hyperedge.
- Δ is the super-structure (optional in some definitions) which captures the relationships or higher-order interactions between the hyperedges themselves (e.g., higher-level connections or interactions between hyperedges).

Example 2:

Let’s define a fuzzy super hypergraph in the context of a social network where we want to model friendship relationships and group memberships.

1. Vertices (V): These represent individuals in the network, e.g., $V = \{A,B,C,D,E\}$.
2. Hyperedges (E): Hyperedges represent groups or relationships between individuals, e.g., friendships or club memberships.

Suppose $E = \{G_1, G_2, G_3\}$, where:

- G_1 is a friendship group (e.g., people who interact regularly),
- G_2 is a club group (e.g., people who belong to a sports club),
- G_3 represents a professional network (e.g., coworkers).

3. Fuzzy Membership Function (μ):

   The membership function μ assigns a degree of membership between 0 and 1 to each vertex for each hyperedge. This reflects how strongly each individual belongs to each group.

   For instance:

   - $\mu(A,G_1)=0.8$ (indicating A is strongly part of friendship group G_1),
   - $\mu(B,G_1)=0.6$ (indicating B is moderately part of friendship group G_1),
   - $\mu(C,G_2)=1.0$ (indicating C fully belongs to club G_2),
   - $\mu(D,G_3)=0.4$ (indicating D has weak membership in the professional network G_3),
   - $\mu(E,G_3)=0.9$ (indicating E has strong membership in G_3).

4. Super-Structure (Δ) (optional):

   - The super-structure captures higher-order relationships between the hyperedges themselves. For instance, it may define a relationship where some groups are interconnected (e.g., members of G_1 might also belong to G_3, forming a meta-relationship).
   - An example could be that if a person belongs to both G_1(friendship group) and G_2 (club group), they also have a membership in a meta-group G_4, which is defined by the relationship between G_1 and G_2.

Visualization of the Example:

Consider the following fuzzy memberships:

Vertex	G_1	G_2	G_3
A	0.8	0.2	0.1
B	0.6	0.4	0.2
C	0.1	1.0	0.3
D	0.3	0.5	0.4
E	0.7	0.3	0.9

Table 1. Fuzzy membership

Table 1 represents the fuzzy memberships where each vertex has a degree of membership in different hyperedges (social groups).

Properties of Fuzzy Super Hypergraphs:

- Fuzzy Inclusion: Unlike traditional hypergraphs, fuzzy super hypergraphs allow varying degrees of membership for vertices in hyperedges, creating a more nuanced representation of relationships.
- Higher-Order Relations: The super-structure can represent complex, higher-order relations between hyperedges themselves, allowing for sophisticated modeling of interdependencies (e.g., a "meta-group" formed by intersections between friendship and club memberships).

- Flexibility: Fuzzy super hypergraphs are flexible and can be used to model various real-world systems, such as social networks, knowledge representation, biological networks, and collaborative systems.

Fuzzy Super Topology

A fuzzy super topology is an advanced concept in topology that combines both fuzzy set theory and superstructures such as hypergraphs or higher-order relational systems. A fuzzy super topology extends traditional fuzzy topologies by introducing fuzzy relationships between fuzzy open sets, creating a more intricate model that can handle complex, dynamic systems with overlapping structures. In a fuzzy super topology, the typical elements of a topology (i.e., open sets) are extended to consider relationships between them, often based on a fuzzy relation that dictates how one fuzzy set or open set relates to another. These relationships are often governed by a fuzzy relation matrix or a super-structure, which adds an extra layer of complexity compared to a standard fuzzy topology. This concept finds applications in areas such as social networks, biological systems, image processing, and recommendation systems, where traditional topological structures may not be sufficient.

Definition 3: Let X be a set and μ: $X \times P(X) \rightarrow [0,1]$ be a fuzzy membership function**,** where P(X) is the power set of X (i.e., the set of all subsets of X).

A fuzzy super topology on X is defined by the following:

1. Fuzzy Open Sets: A fuzzy open set $U \subseteq X$ is a fuzzy subset of X, represented by a membership function μ_U: $X \rightarrow [0,1]$. The degree of membership of an element $x \in X$ in the fuzzy set U is denoted as $\mu_U(x)$.
2. Super-Structure of Relations: A super-structure Δ defines the fuzzy relationships between different fuzzy open sets in P(X). This can be described by a fuzzy relation matrix Δ, where each entry $\Delta(U,V)$ represents the fuzzy degree of "relationship" or "overlap" between the fuzzy open sets U and V. This fuzzy relation captures how fuzzy open sets intersect, relate, or influence each other.

Conditions for a Fuzzy Super Topology:

For a collection T_f of fuzzy open sets to be considered a fuzzy super topology on X, the following conditions must be satisfied:

1. Fuzzy Closure:
    - For any fuzzy open set $U \in T_f$, the fuzzy complement U^c (the complement of U) must also belong to T_f, and the degree of membership of a point x in the complement is given by $\mu_{U^c}(x) = 1 - \mu_U(x)$.
2. Union of Fuzzy Open Sets:
    - The fuzzy union of a family of fuzzy open sets $\{U_i\}$ should be a fuzzy open set. The degree of membership of x in the union is the maximum of the degrees of membership in the individual fuzzy open sets.
3. Intersection of Fuzzy Open Sets:
    - The fuzzy intersection of a family of fuzzy open sets $\{U_i\}$ should also be a fuzzy open set. The degree of membership of x in the intersection is the minimum of the degrees of membership in the individual fuzzy open sets.
4. Super-Structure of Fuzzy Relations:
    - For every pair of fuzzy open sets U, $V \in T_f$, there should exist a fuzzy relationship $\Delta(U,V)$ that satisfies certain conditions, such as:
        - $\Delta(U,V) \geq \min(\mu_{U(x)}, \mu_{V(x)})$ for all $x \in X$.
        - The relation Δ should be reflexive, i.e., $\Delta(U,U) = 1$ for all $U \in T_f$.

Example 4:

Let X = {1,2,3,4} and let's define a fuzzy topology on X by considering fuzzy open sets.

Fuzzy Membership Functions:
Let the fuzzy open sets U,V and W on X be represented by the following membership functions:

- $\mu_{U(1)} = 0.8$, $\mu_{U(2)} = 0.6$, $\mu_{U(3)} = 0.3$, $\mu_{U(4)} = 0.1$
- $\mu_{V(1)} = 0.2$, $\mu_{V(2)} = 0.5$, $\mu_{V(3)} = 0.7$, $\mu_{V(4)} = 0.9$
- $\mu_{W(1)} = 0.4$, $\mu_{W(2)} = 0.7$, $\mu_{W(3)} = 0.9$, $\mu_{W(4)} = 0.2$

The fuzzy open sets U, V and W represent subsets of X, where each element in X has a degree of membership in each fuzzy set.
Super-Structure (Fuzzy Relations):
We define the fuzzy relation matrix Δ between the fuzzy open sets U, V and W as follows, where Δ(U,V) represents the degree of overlap or relation between U and V:

Relation	U	V	W
U	1.0	0.6	0.4
V	0.6	1.0	0.7
W	0.4	0.7	1.0

Table 2. Relation between open sets

This matrix (Table 2) defines how the fuzzy open sets relate to each other. For example, Δ(U,V) = 0.6 indicates that the relationship between U and V is of medium strength, while Δ(U,W) = 0.4 indicates a weaker relationship between U and W.

Fuzzy Coloring in a Fuzzy Super Hypergraph:

The fuzzy coloring concept is an extension of the traditional graph coloring, where the idea of colors (or labels) is generalized to fuzzy values, allowing for more nuanced relationships between vertices. Instead of assigning a specific color to a vertex, a fuzzy coloring assigns a degree of membership to each possible color for every vertex. A fuzzy coloring in a fuzzy super hypergraph is an assignment of fuzzy values to the vertices in the vertex set of the fuzzy super hypergraph such that each vertex is assigned a "color" with a degree of membership, which is represented by a fuzzy membership function.

Definition 5: Let H = (V,E,μ,Δ) be a fuzzy super hypergraph. A fuzzy coloring of the fuzzy super hypergraph is a function $\gamma: V \rightarrow [0,1]^C$ where:

- C is the set of all possible colors, and
- γ(v) is a fuzzy set of colors assigned to the vertex v, where each color c ∈ C is associated with a degree of membership $\mu_\gamma(v,c) \in [0,1]$.

For every vertex $v_i \in V$, the fuzzy coloring $\gamma(v_i)$ is a fuzzy set on the set of colors, indicating the degree of "membership" or "suitability" of each color for vertex v_i. In other words, the vertex v_i does not get a single color but rather a fuzzy set of colors with associated membership degrees.

Example 6:
Consider the following fuzzy super hypergraph H = (V,E,μ,Δ) with:

- Vertex set $V = \{v_1,v_2,v_3\}$,
- Hyperedge set $E = \{G_1,G_2\}$,

- Fuzzy membership function μ defined as in Table 3:

Vertex	G_1	G_2
v_1	0.8	0.5
v_2	0.6	0.7
v_3	0.2	0.9

Table 3. Fuzzy Membership

- The super-structure relation $\Delta(G_1,G_2) = 0.8$.

Now, suppose we want to assign a fuzzy coloring to the vertices, where the set of colors is $C = \{c_1,c_2\}$. The fuzzy coloring function γ can be defined as follows:

- For v_1: $\gamma(v_1) = \{(c_1,0.7),(c_2,0.3)\}$, meaning that vertex v_1 has a degree of 0.7 for color c_1 and 0.3 for color c_2.
- For v_2: $\gamma(v_2) = \{(c_1,0.4),(c_2,0.6)\}$, meaning that vertex v_2 has a degree of 0.4 for color c_1 and 0.6 for color c_2.
- For v_3: $\gamma(v_3) = \{(c_1,0.2),(c_2,0.8)\}$, meaning that vertex v_3 has a degree of 0.2 for color c_1 and 0.8 for color c_2.

Here, the vertices do not have a single color, but rather, each vertex is assigned a fuzzy set of colors, each with a corresponding degree of membership.

This section provides a general framework for fuzzy coloring in fuzzy super hypergraphs, capturing the complex relationships between vertices and hyperedges in a fuzzy, flexible manner.

3. Generating Fuzzy Super Topology on Vertex Set of Fuzzy Super Hypergraphs

Technique for Generating Fuzzy Super Topology on Vertex Set of a Fuzzy Super Hypergraph Using Adjacency Relation:

The process of generating a fuzzy super topology on the vertex set of a fuzzy super hypergraph using adjacency relation involves the following steps:

1. Define the fuzzy super hypergraph components.
2. Construct the fuzzy adjacency relation based on the fuzzy hyperedges.
3. Extend the adjacency relation to account for the super-structure (fuzzy relations between hyperedges).
4. Generate fuzzy open sets that form a fuzzy super topology.

Step-by-Step Approach:

1. Components of a Fuzzy Super Hypergraph

Let a fuzzy super hypergraph H be defined as $H = (V,E,\mu,\Delta)$ Where:

- V is the set of vertices.

- E is the set of hyperedges.
- μ: $V \times E \rightarrow [0,1]$ is the fuzzy membership function that gives the degree of membership of each vertex to each hyperedge.
- Δ: $E \times E \rightarrow [0,1]$ is the super-structure relation that captures the fuzzy relationships between hyperedges.

2. Define the Fuzzy Adjacency Relation

The fuzzy adjacency relation $R(v_i,v_j)$ for any two vertices $v_i,v_j \in V$ is computed based on the fuzzy membership values of the vertices in the hyperedges. The adjacency relation reflects the strength of their connection, determined by the fuzzy memberships in the hyperedges they belong to.

The fuzzy adjacency relation between two vertices v_i and v_j is given by:

$R(v_i,vj) = \max_{Gk \in E} \min(\mu(v_i,G_k),\mu(v_j,G_k))$

This means that the adjacency relation $R(v_i,v_j)$ is determined by the maximum degree of overlap in the fuzzy memberships of v_i and v_j across all the hyperedges G_k.

3. Extend the Adjacency Relation with Super-Structure Δ

Now, we extend the adjacency relation to account for the relationships between hyperedges themselves using the super-structure relation Δ. This super-structure relation defines how strongly two hyperedges G_k and G_m are related.

The extended fuzzy adjacency relation between two vertices v_i and v_j can be written as:

$R_s(v_i,v_j) = \max_{Gk,Gm \in E} \min(\mu(v_i,G_k),\mu(v_j,G_m),\Delta(G_k,G_m))$.

Here:

- $R_s(v_i,v_j)$ is the extended adjacency relation considering the super-structure.
- $\Delta(G_k,G_m)$ is the degree of relation between hyperedges G_k and G_m.

4. Generate Fuzzy Open Sets

Next, we generate fuzzy open sets from the fuzzy adjacency relations. The idea is to select sets of vertices that are "adjacent" to each other above a certain fuzzy threshold.

To define a fuzzy open set U, we select vertices that are sufficiently "adjacent" to each other. This can be done using a threshold α to determine which vertices are sufficiently connected.

Let U_α be a fuzzy open set defined by the relation:

$U_\alpha = \{v \in V: R_s(v,v_j) \geq \alpha, \forall\ v_j \in U_\alpha\}$

This defines a fuzzy open set based on the adjacency relation R_s where vertices are considered adjacent to the open set U_α if their adjacency relation exceeds the threshold α.

By following this technique, we can generate a fuzzy super topology on the vertex set of a fuzzy super hypergraph using adjacency relations and a super-structure relation. The adjacency relations define how vertices

are connected, while the super-structure relations account for the relationships between hyperedges. This method enables the creation of fuzzy topological spaces with complex, dynamic connections, useful in areas like network analysis, social dynamics, and biological modeling.

Example 7:

Consider the following fuzzy super hypergraph:

- Vertex set $V = \{v_1,v_2,v_3\}$
- Hyperedges $E = \{G_1,G_2\}$

Fuzzy Memberships as in Table 4:

Vertex	G_1	G_2
v_1	0.8	0.5
v_2	0.6	0.7
v_3	0.2	0.9

Table 4. Fuzzy Membership

Super-structure Relation Δ: $\Delta(G_1,G_2) = 0.8$
This means that the relationship between hyperedges G_1 and G_2 is 0.8 (a strong relation).

Step 1: Compute the Fuzzy Adjacency Relation

- $R(v_1,v_2)=\max(\min(0.8,0.6),\min(0.5,0.7))=\max(0.6,0.5)=0.6$
  $R(v_1,v_3)=\max(\min(0.8,0.2),\min(0.5,0.9))=\max(0.2,0.5)=0.5$
  $R(v_2,v_3)=\max(\min(0.6,0.2),\min(0.7,0.9))=\max(0.2,0.7)=0.7$

Step 2: Compute the Extended Fuzzy Adjacency Relation

Using the super-structure relation $\Delta(G_1,G_2) = 0.8$, we extend the adjacency relation:

- $R_s(v_1,v_2)=\max(\min(0.8,0.6,0.8),\min(0.5,0.7,0.8))=\max(0.6,0.5)=0.6$
  $R_s(v_1,v_3)=\max(\min(0.8,0.2,0.8),\min(0.5,0.9,0.8))=\max(0.2,0.5)=0.5$
  $R_s(v_2,v_3)=\max(\min(0.6,0.2,0.8),\min(0.7,0.9,0.8))=\max(0.2,0.7)=0.7$

Step 3: Generate Fuzzy Open Sets

For a threshold $\alpha = 0.6$, we generate fuzzy open sets. Any vertex v that has an adjacency relation greater than or equal to 0.6 with another vertex will belong to the fuzzy open set. For example:

- $R_s(v_1,v_2) = 0.6 = 0.6$, so v_1 and v_2 are in the same fuzzy open set.
- $R_s(v_1,v_3) = 0.5 < 0.6$, so v_1 and v_3 are not in the same fuzzy open set.

Thus, the fuzzy open set corresponding to this threshold is $\{v_1,v_2\}$.

Theorem 8: The technique of generating a Fuzzy Super Topology on the vertex set of a Fuzzy Super Hypergraph using the adjacency relation generates a Fuzzy Super Topology

Proof: We aim to prove that the collection of fuzzy open sets generated by this adjacency relation forms a fuzzy super topology.

Conditions for a Fuzzy Super Topology
A fuzzy super topology on the vertex set V must satisfy the following conditions:

1. The complement of any fuzzy open set is a fuzzy open set.
2. The union of any collection of fuzzy open sets is a fuzzy open set.
3. The intersection of any collection of fuzzy open sets is a fuzzy open set.
4. The fuzzy relations between fuzzy open sets satisfy the conditions for a super topology.

Condition 1:
Let U be a fuzzy open set generated by the adjacency relation with degree $\mu_{U(v)}$ for each vertex $v \in V$. The complement of U, denoted U^c, consists of all vertices whose degree of membership in U is less than 1. The degree of membership of a vertex v in the complement U^c is given by $\mu_{U^c}(v) = 1-\mu_U(v)$. Since the membership function $\mu_U(v)$ satisfies the conditions for fuzzy sets (i.e., it takes values in the range [0,1]), it follows that the complement U^c is also a fuzzy set, with the degree of membership $1 - \mu_U(v)$. Thus, the complement of a fuzzy open set is also a fuzzy open set.
Condition 2:
Let $\{U_i\}$, $i \in I$ be a collection of fuzzy open sets, with $\mu_{Ui}(v)$ representing the degree of membership of each vertex v in the fuzzy open set U_i. The union of these fuzzy open sets is a fuzzy open set, and the degree of membership of vertex v in the union $U_\cup$ is given by $\mu U_\cup(v) = \max_{i\in I} \mu_{Ui}(v)$. This means that the degree of membership of vertex v in the union is the maximum degree of membership in any of the sets U_i. Since the maximum of fuzzy membership values is also a fuzzy membership value (it lies within the range [0,1]), the union of fuzzy open sets is a fuzzy open set. Thus, the union of any collection of fuzzy open sets is a fuzzy open set.
Condition 3:
Let $\{U_i\}$, $i \in I$ be a collection of fuzzy open sets. The intersection of these fuzzy open sets is a fuzzy open set, and the degree of membership of vertex v in the intersection $U_\cap$ is given by $\mu_{U\cap}(v) = \min_{i\in I} \mu_{Ui}(v)$. This means that the degree of membership of vertex v in the intersection is the minimum of the membership values in the individual fuzzy open sets. Since the minimum of fuzzy membership values is also a fuzzy membership value (it lies within the range [0,1]), the intersection of fuzzy open sets is a fuzzy open set. Thus, the intersection of any collection of fuzzy open sets is a fuzzy open set.
Condition 4:
The super-structure relation Δ between hyperedges defines the fuzzy relations between fuzzy open sets in the fuzzy super topology. The adjacency relations $R(v_i,v_j)$ are extended to include these fuzzy relations by incorporating the super-structure Δ between hyperedges, leading to the extended fuzzy adjacency relation $R_s(v_i,v_j) = \max_{Gk,Gl\in E} \min(\mu(v_i,G_k), \mu(v_j,G_l), \Delta(G_k,G_l))$.

For this relation to be a valid fuzzy relation, it must satisfy the following properties:

1. Reflexivity: This follows from the fact that $\mu(v_i,G_k)$ represents the degree to which v_i belongs to hyperedge G_k, and $\Delta(G_k,G_k) = 1$ (since the super-structure relation is reflexive).
2. Symmetry: This follows from the fact that the adjacency relation R_s is symmetric in its arguments, and the super-structure relation $\Delta(G_k,G_l) = \Delta(G_l,G_k)$ is symmetric.
3. Transitivity: The fuzzy relation $R_s(v_i,v_j)$ is transitive if for any three vertices $v_i,v_j,v_k \in V$, the condition $R_s(v_i,v_j) \geq \alpha$ and $R_s(v_j,v_k) \geq \alpha$ implies $R_s(v_i,v_k) \geq \alpha$. This follows from the fact that the maximum and minimum operations involved in the fuzzy relation respect transitivity properties.

Thus, the fuzzy relations between fuzzy open sets generated by the adjacency relation and super-structure relation satisfy the conditions of a fuzzy super topology.

We have shown that the technique of generating a fuzzy super topology on the vertex set of a fuzzy super hypergraph using the adjacency relation and super-structure relation satisfies all the conditions required for a fuzzy super topology. Therefore, the technique of generating a fuzzy super topology using the adjacency relation on the vertex set of a fuzzy super hypergraph generates a valid fuzzy super topology.

Technique for Generating Fuzzy Super Topology on the Vertex Set of a Fuzzy Super Hypergraph Using Fuzzy Coloring:

We can generate a fuzzy super topology on the vertex set V of a fuzzy super hypergraph $H = (V,E,\mu,\Delta)$ using fuzzy coloring by utilizing the fuzzy coloring assignments as the basis for defining fuzzy open sets in the topological space. The fuzzy coloring can provide the fuzzy relations between vertices, which in turn can be used to define the open sets that satisfy the conditions for a fuzzy super topology.

Steps for Generating the Fuzzy Super Topology

1. Fuzzy Coloring Assignment:
    - Given a fuzzy super hypergraph $H = (V,E,\mu,\Delta)$ where $V = \{v_1,v_2,\ldots,v_n\}$ is the set of vertices and $C = \{c_1,c_2,\ldots,c_k\}$ is the set of possible colors, we define a fuzzy coloring function $\gamma: V \rightarrow [0,1]^C$ that assigns to each vertex a fuzzy set of colors, where each color c_i has a degree of membership $\mu_\gamma(v,c_i)$ indicating the degree of association of vertex v with color c_i.
2. Define Fuzzy Open Sets Using Fuzzy Coloring:
    - A fuzzy open set is generated by selecting a threshold value $\theta \in [0,1]$ for the degree of membership. The vertices that have a membership degree $\mu_\gamma(v,c_i) \geq \theta$ for some color c_i will be included in a fuzzy open set.

    Specifically, a fuzzy open set $O \subseteq V$ corresponding to color c_i is defined as :

    $O_{ci,\theta} = \{v \in V: \mu_\gamma(v,c_i) \geq \theta\}$

    This means that a vertex is included in the fuzzy open set for color c_i if its degree of membership to that color is above the threshold θ. We can form multiple fuzzy open sets for different colors and thresholds.

3. Incorporate Super-Structure Relation Δ:
    - The fuzzy super topology must take into account the fuzzy relationship between hyperedges, as represented by the super-structure relation Δ, which defines the degree of association between different hyperedges.
    - The fuzzy open sets are also influenced by the super-structure relation between hyperedges, which reflects the proximity of vertices across hyperedges. This relation can be used to fine-tune the membership values, making sure the fuzzy open sets respect the fuzzy relationships between vertices defined by μ and Δ.
4. Construct Fuzzy Super Topology:
    - The collection of fuzzy open sets forms the fuzzy super topology on the vertex set V. The fuzzy open sets must satisfy the following properties:
        1. The complement of a fuzzy open set is a fuzzy open set.
        2. The union of fuzzy open sets is a fuzzy open set.
        3. The intersection of fuzzy open sets is a fuzzy open set.

The technique of generating a fuzzy super topology on the vertex set of a fuzzy super hypergraph using fuzzy coloring involves defining fuzzy open sets based on the fuzzy membership values assigned to the vertices for different colors. The super-structure relation between hyperedges adds an additional layer of fuzzy

relationships that influence the construction of the topology. This technique can be used to model complex relationships in networks, biological systems, and other fields requiring fuzzy structures.

Example 9:

Let us consider a simple fuzzy super hypergraph:

- Vertex set $V = \{v_1,v_2,v_3,v_4\}$,
- Hyperedge set $E = \{G_1,G_2\}$,
- Fuzzy membership function $\mu(v_i,G_j)$ is given in Table 5:

Vertex	G_1	G_2
v_1	0.8	0.6
v_2	0.5	0.9
v_3	0.7	0.4
v_4	0.3	0.7

Table 5. Fuzzy Membership

- Super-structure relation $\Delta(G_1,G_2) = 0.5$, meaning the fuzzy relation between hyperedges G_1 and G_2 has a membership degree of 0.5.
- Let the set of colors $C = \{c_1,c_2\}$.

Now, suppose we define the fuzzy coloring function γ as follows:

- $\gamma(v_1) = \{(c_1,0.7),(c_2,0.3)\}$,
- $\gamma(v_2) = \{(c_1,0.5),(c_2,0.8)\}$,
- $\gamma(v_3) = \{(c_1,0.6),(c_2,0.4)\}$,
- $\gamma(v_4) = \{(c_1,0.3),(c_2,0.7)\}$.

Step-by-Step Construction of the Fuzzy Super Topology:

1. Fuzzy Open Sets for Color c_1:
    - For a threshold $\theta = 0.5$, the fuzzy open set corresponding to color c_1 is:

    $O_{c1,0.5} = \{v_1,v_2,v_3\}$, $(\mu_\gamma(v_1,c_1) = 0.7, \mu_\gamma(v_2,c_1) = 0.5, \mu_\gamma(v_3,c_1) = 0.6)$
2. Fuzzy Open Sets for Color c_2:
    - For a threshold $\theta = 0.5$, the fuzzy open set corresponding to color c_2 is:

    $O_{c2,0.5} = \{v_2,v_4\}$, $(\mu_\gamma(v_2,c_2) = 0.8, \mu_\gamma(v_4,c_2) = 0.7)$
3. Adjust with the Super-Structure Relation:
    - The super-structure relation $\Delta(G_1,G_2) = 0.5$ influences how we adjust the membership values for vertices across hyperedges. This relationship modifies the overall structure of the fuzzy open sets to respect the fuzzy adjacency of vertices across hyperedges. For example, if vertices v_1 and v_2 are highly related via G_2, the membership of v_1 in color c_2 might be adjusted to reflect this relationship.
4. Final Fuzzy Super Topology:
    - The fuzzy open sets generated for different colors and thresholds combine to form the fuzzy super topology on V. For example, we could define:

    $O_\cup = O_{c1,0.5} \cup O_{c2,0.5} = \{v_1,v_2,v_3,v_4\}$

    The union and intersection of these sets also form valid fuzzy open sets, maintaining the properties required for a fuzzy super topology.

Theorem 10: The technique of generating Fuzzy Super Topology on the Vertex Set of Fuzzy Super Hypergraph using Fuzzy Coloring generates a Fuzzy Super Topology

Proof: We need to show that the collection of fuzzy open sets generated by fuzzy coloring forms a fuzzy super topology on V, which must satisfy the following conditions:

1. The complement of any fuzzy open set is a fuzzy open set.
2. The union of any collection of fuzzy open sets is a fuzzy open set.
3. The intersection of any collection of fuzzy open sets is a fuzzy open set.
4. The fuzzy relations between fuzzy open sets must satisfy the properties of a super topology (i.e., they must respect the fuzzy relations induced by the super-structure relation Δ.

Condition 1:

Let $O_{ci,\theta}$ be a fuzzy open set corresponding to color c_i with threshold θ, i.e.,

$O_{ci,\theta} = \{v \in V: \mu_\gamma(v,c_i) \geq \theta\}$

The complement of this fuzzy open set is the set of vertices that do not satisfy the condition:

$Oc_i,\theta^c = \{v \in V: \mu_\gamma(v,ci) < \theta\}$

The membership degree of a vertex v in Oc_i,θ^c is $1 - \mu_\gamma(v,c_i)$, which is still a fuzzy set, as it lies within the range [0,1]. Therefore, the complement of a fuzzy open set is also a fuzzy open set.

Condition 2:

Let $\{O_{ci}, \theta_i\}$ $i \in I$ be a collection of fuzzy open sets, where each $O_{ci}, \theta_i = \{v \in V | \mu_\gamma(v,c_i) \geq \theta_i\}$ for each $i \in I$. The union of these fuzzy open sets is the set of vertices that belong to at least one of these sets:

$O_\cup = \bigcup_{i \in I} O_{ci,\theta i} = \{v \in V : \text{there exists } i \in I, \mu_\gamma(v,c_i) \geq \theta_i\}$

The degree of membership of a vertex v in this union is $\mu_{O\cup}(v) = \max_{i \in I} \mu_\gamma(v,c_i)$

Since the maximum of fuzzy membership values is still a fuzzy membership value in the range [0,1], the union of any collection of fuzzy open sets is itself a fuzzy open set.

Condition 3:

Let $\{O_{ci}, \theta_i\}$, $i \in I$ be a collection of fuzzy open sets, where each $O_{ci},\theta_i = \{v \in V: \mu_\gamma(v,c_i) \geq \theta_i\}$. The intersection of these fuzzy open sets is the set of vertices that belong to all of these sets: $O_\cap = \bigcap_{i \in I} O_{ci},\theta_i = \{v \in V : \forall i \in I, \mu_\gamma(v,c_i) \geq \theta_i\}$.

The degree of membership of a vertex v in this intersection is $\mu_{O\cap}(v) = \min_{i \in I} \mu_\gamma(v,c_i)$

Since the minimum of fuzzy membership values is also a fuzzy membership value in the range [0,1], the intersection of any collection of fuzzy open sets is itself a fuzzy open set.

Condition 4:

The fuzzy relations between fuzzy open sets are defined based on the fuzzy coloring of vertices, which are influenced by the fuzzy membership function μ and the super-structure relation Δ between hyperedges. These relations must satisfy the properties of a super topology, namely:

1. Reflexivity: Each fuzzy open set must include the vertex itself, i.e., $v \in O_{ci}, \theta$ for some c_i and θ, because $\mu_\gamma(v,c_i) \geq 0$ by definition of fuzzy membership.
2. Symmetry: The fuzzy relation between two vertices v_1 and v_2 is symmetric because the fuzzy coloring function γ assigns symmetric degrees of membership to both vertices with respect to each color c_i. Specifically, $\mu_\gamma(v_1,c_i) = \mu_\gamma(v_2,c_i)$ ensures that the fuzzy relation between any two vertices is symmetric.
3. Transitivity: If two vertices v_1 and v_2 are in the same fuzzy open set, and v_2 and v_3 are in the same fuzzy open set, then v_1 and v_3 must also be in the same fuzzy open set, with the degree of membership determined by the intersection (i.e., the minimum) of the fuzzy sets. This transitivity is ensured by the fuzzy intersection operation defined above.

Since the fuzzy open sets respect these properties of reflexivity, symmetry, and transitivity, the fuzzy relations between fuzzy open sets satisfy the conditions for a super topology.

Thus, the technique of generating a fuzzy super topology using fuzzy coloring on the vertex set of a fuzzy super hypergraph generates a valid fuzzy super topology.

4. Comparison of Techniques

In this comparison, we explore two distinct methods for generating a fuzzy super topology on the vertex set V of a fuzzy super hypergraph $H = (V,E,\mu,\Delta)$. These techniques are based on:

1. Adjacency Relation.
2. Fuzzy Coloring.

Each method has its own approach to constructing the fuzzy open sets, and we will evaluate these approaches based on several key aspects:

Topological Properties:

Adjacency Relation-Based Topology:

- Open Sets: The open sets are constructed by considering the adjacency relations between vertices, so vertices that are "close" or connected by hyperedges are grouped together.
- Neighborhoods: Vertices that are adjacent to each other (as defined by the fuzzy adjacency relation) will belong to the same fuzzy open set. This topology reflects the global connectivity between vertices and can capture the structural relationships within the hypergraph.
- Influence of Hyperedges: The adjacency relation directly reflects the structure of the hypergraph, where vertices that are close in terms of hyperedges will belong to the same fuzzy open set. This method can be more robust when the focus is on the topological structure of the hypergraph.

Fuzzy Coloring-Based Topology:

- Open Sets: The open sets are based on fuzzy membership values to colors, forming regions of the vertex set that exhibit similar characteristics (in terms of color membership).
- Neighborhoods: Vertices in the same fuzzy open set are similar in terms of their fuzzy color membership. This method provides a way to group vertices by their local properties and relationships to colors.
- Flexibility: The threshold θ can be adjusted to form more or less granular fuzzy open sets. This allows for a fine-tuned control over how "tight" or "loose" the topology is around each vertex based on its membership to a color.

Applicability and Use Cases:

Fuzzy Coloring:

- Local Vertex Properties: Ideal for situations where we are interested in local attributes of vertices, such as their degree of association with certain colors or classes.
- Color-Coding Problems: Useful in scenarios where vertices need to be classified or grouped based on fuzzy attributes (e.g., community detection, classification in social networks, etc.).
- Dynamic Color Adjustments: If the coloring can be adjusted dynamically, this technique can provide a flexible way to evolve topologies over time as the membership values change.

Adjacency Relation:

- Global Network Structure: Well-suited for scenarios that focus on the global connectivity of the hypergraph. This includes problems where the relationships between vertices are crucial (e.g., communication networks, biological networks, etc.).
- Connectivity and Path finding: Useful for applications where the closeness or connectivity between vertices is important, such as path finding algorithms or clustering based on adjacency.

Table 6 describes the key differences between the methods.

Feature	Adjacency Relation	Fuzzy Coloring
Primary Basis	Relies on the fuzzy adjacency relations between vertices.	Relies on the fuzzy assignment of colors to vertices.
Conceptual Focus	Focuses on the global connectivity between vertices based on their adjacency through hyperedges.	Focuses on the local property of how vertices are associated with colors.
Open Set Construction	Fuzzy open sets are created based on the adjacency degree between vertices.	Fuzzy open sets are created based on color membership values and thresholds.
Nature of Relations	The relations are between vertices, reflecting how connected they are through hyperedges.	The relations are between vertices and colors.
Influence of Hyperedges	The influence of hyperedges is direct, as adjacency is determined by hyperedge participation.	The influence of hyperedges is indirect, affecting vertex membership to colors.
Sensitivity to Vertex Properties	Sensitive to how strongly vertices are connected (adjacent) to each other.	Sensitive to how vertices are "colored" in relation to their membership degrees for each color.
Complexity	May be more complex as it involves calculating adjacency relations and connectivity between vertices.	May be simpler if fewer colors are considered, focusing on the local characteristics of vertices.
Flexibility	Less flexible, depends on adjacency structure	High, easily adjustable thresholds

Table 6. Key Differences

Both techniques for generating a fuzzy super topology offer unique advantages. The adjacency relation method, on the other hand, is better suited for capturing global structure and connectivity between vertices, reflecting the inherent relationships in the hypergraph. The fuzzy coloring method excels in handling local properties and provides flexibility in defining vertex relationships through color membership. The choice between the two depends on the specific problem being addressed and whether the focus is on local or global characteristics of the fuzzy super hypergraph.

5. **Conclusion**

This chapter presents a comparative study of fuzzy super topologies generated on the vertex set of a fuzzy super hypergraph using adjacency and fuzzy super coloring. By integrating concepts from fuzzy set theory, hypergraph topology, and graph coloring, the study explores different topological structures that emerge from these methodologies.

The key findings indicate that topologies generated using adjacency relations exhibit a stronger dependence on the underlying connectivity structure of the fuzzy super hypergraph. These topologies capture the local and global connectivity properties of vertices based on their direct and indirect associations within hyperedges. On the other hand, topologies induced by fuzzy super coloring introduce a different perspective by grouping vertices based on their color classes, which reflect similarity in terms of membership functions and structural roles in the hypergraph.

This comparative analysis highlights that adjacency-based fuzzy super topologies tend to be more flexible in handling variations in hyperedge configurations, making them useful for studying network reachability and clustering properties. Meanwhile, fuzzy super coloring-based topologies provide a categorical classification that can be applied in resource allocation, optimization, and decision-making problems where partitioning of the hypergraph is required.

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Chapter 6

A Concise Review on Various Concepts of SuperHyperStructures

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Abstract

A *Hyperstructure* builds on the notion of a powerset, offering a framework to model relationships among elements within a set. Extending this idea, a *SuperHyperStructure* employs the n-th powerset to capture multi-layered hierarchical systems, enabling deeper abstractions and greater complexity. Here, the term *Structure* refers broadly to any mathematical or real-world concept. Representative notions developed within this framework include superhypergraphs, superhypertopology, and superhypersoft sets. Since the study of SuperHyperStructures has only recently begun, this paper aims to contribute to their dissemination by introducing and examining a variety of SuperHyperStructure-based concepts, emphasizing their extensions and potential applications.

Keywords: Hyperstructure, Superhyperstructure, Powerset, nth Powerset

1 Introduction

1.1 Hyperstructures and Superhyperstructures

Throughout this paper, the term *Structure* is used broadly to denote any mathematical construct or real-world entity. Many such entities admit a natural hierarchy, a viewpoint that motivates recent work on *hyperstructures* and *superhyperstructures*. A *Hyperstructure* enlarges the classical powerset perspective, thereby extending its reach across a wide spectrum of mathematical systems and abstract settings [1–4]. Advancing this idea, a *Superhyperstructure* leverages the n-fold iterated powerset to support layered modeling and stepwise abstraction; in turn, it offers a principled scaffold for expressing ever more intricate relationships and deeper structural levels [4, 5]. The rising prominence of SuperHyperStructure theory has further inspired companion notions, including the *SuperHyperGraph* [6, 7] and the *SuperHyperFunction* [8, 9]. Collectively, these frameworks facilitate hierarchical and multilevel modeling, capture higher-order relations, integrate heterogeneous data, enable compositional abstractions with richer forms of uncertainty, and scale to complex, multi-agent, multi-criteria settings.

1.2 Our Contributions

Against this backdrop, the study of hyperstructures and superhyperstructures is both timely and consequential. Because a systematic theory of SuperHyperStructures is still emergent, our goal is to aid its development by presenting and analyzing a range of SuperHyperStructure-based concepts, with particular attention to their extensions and prospective applications. We hope that the material compiled here will help readers consolidate their understanding and stimulate further research on hyperstructures and superhyperstructures.

2 Preliminaries

Before introducing various concepts, we begin with some fundamental definitions. A *Hyperstructure* is based on the concept of a powerset, providing a framework to model relationships among elements within a set [2, 10, 11]. Extending this concept, a *SuperHyperStructure* employs the n-th powerset to represent systems with multi-layered hierarchical relationships, enabling deeper abstractions and increased complexity [4, 5]. It should be noted that such concepts are typically considered within the scope of finite structures rather than infinite ones.

2.1 Powerset and nth-Powetset

The *powerset* of a set is the collection of all its subsets, including the empty set and the set itself. The *n-th powerset* is constructed iteratively by applying the powerset operation n times, generating higher-order layers of subsets of subsets. Below, we formally define the n-th powerset as a foundation for these structures.

Definition 2.1 (Base Set)**.** A *base set* S is the fundamental set from which more complex mathematical constructs, such as powersets and hyperstructures, are derived. It is formally defined as

$$S = \{\, x \mid x \text{ is an element within a specified domain} \,\}.$$

Every element of higher-order structures like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originates from the elements of S.

Definition 2.2 (Powerset)**.** [12] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all subsets of S, including both the empty set and the set S itself. Formally,

$$\mathcal{P}(S) = \{\, A \mid A \subseteq S \,\}.$$

Definition 2.3 (n-th Powerset)**.** (cf. [4, 13])

The n-th powerset of a set H, denoted $\mathcal{P}_n(H)$, is defined recursively, beginning with the standard powerset. The construction is given by

$$\mathcal{P}_1(H) = \mathcal{P}(H), \qquad \mathcal{P}_{n+1}(H) = \mathcal{P}\big(\mathcal{P}_n(H)\big), \qquad n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $\mathcal{P}^*_n(H)$, is defined by

$$\mathcal{P}^*_1(H) = \mathcal{P}^*(H), \qquad \mathcal{P}^*_{n+1}(H) = \mathcal{P}^*\big(\mathcal{P}^*_n(H)\big),$$

where $\mathcal{P}^*(H)$ denotes the powerset of H with the empty set excluded.

Example 2.4 (Retail baskets and promotions as elements of $\mathcal{P}_2(H)$)**.** Let the item universe be $H = \{\text{Milk}, \text{Bread}, \text{Apples}\}$. A single shopping basket is any subset of H, so baskets live in the powerset

$$\begin{aligned}\mathcal{P}(H) = \big\{&\varnothing, \{\text{Milk}\}, \{\text{Bread}\}, \{\text{Apples}\},\\ &\{\text{Milk}, \text{Bread}\}, \{\text{Milk}, \text{Apples}\},\\ &\{\text{Bread}, \text{Apples}\}, \{\text{Milk}, \text{Bread}, \text{Apples}\}\big\}.\end{aligned}$$

A *promotion* can be modeled as a set of eligible baskets, i.e., an element of

$$\mathcal{P}\big(\mathcal{P}(H)\big) = \mathcal{P}_2(H)$$

. For instance,

$$\text{Promo}_{\text{any-2}} = \big\{\{\text{Milk}, \text{Bread}\}, \{\text{Milk}, \text{Apples}\}, \{\text{Bread}, \text{Apples}\}\big\} \in \mathcal{P}_2(H),$$

captures "buy any two distinct items." Another promotion,

$$\text{Promo}_{\text{staples}} = \big\{\{\text{Milk}\}, \{\text{Bread}\}, \{\text{Milk}, \text{Bread}\}\big\} \in \mathcal{P}_2(H),$$

collects baskets viewed as "staples." Thus, individual baskets are elements of $\mathcal{P}(H)$, while marketing rules are subsets of $\mathcal{P}(H)$, i.e., elements of $\mathcal{P}_2(H)$.

Example 2.5 (Hospital rostering: shifts in $\mathcal{P}(H)$, weekly plans in $\mathcal{P}_2(H)$)**.** Let

$$H = \{\text{Alice}, \text{Bob}, \text{Carol}, \text{Dan}\}$$

be available nurses. A single shift assignment is a subset of H, hence an element of $\mathcal{P}(H)$. Suppose the hospital considers the following day-shift candidates:

$$S_{\text{Mon}} = \{\text{Alice}, \text{Bob}\},$$

$$S_{\text{Tue}} = \{\text{Alice}, \text{Carol}\},$$

$$S_{\text{Wed}} = \{\text{Bob}, \text{Dan}\}.$$

A tentative weekly plan is then the set of these daily shifts,

$$R_{\text{week}} = \{S_{\text{Mon}}, S_{\text{Tue}}, S_{\text{Wed}}\} \in \mathcal{P}(\mathcal{P}(H)) = \mathcal{P}_2(H).$$

If the scheduling office keeps several alternatives $R^{(1)}_{\text{week}}, R^{(2)}_{\text{week}}, R^{(3)}_{\text{week}} \in \mathcal{P}_2(H)$, then a monthly portfolio $\mathcal{M} = \{R^{(1)}_{\text{week}}, R^{(2)}_{\text{week}}, R^{(3)}_{\text{week}}\}$ is an element of $\mathcal{P}_3(H) = \mathcal{P}(\mathcal{P}_2(H))$. Thus, staffings ($\in \mathcal{P}(H)$), weekly rosters ($\in \mathcal{P}_2(H)$), and portfolios of rosters ($\in \mathcal{P}_3(H)$) form successive real-life uses of the n-th powerset.

For reference, Table 1 provides a concise overview of Set, Powerset, and n-th Powerset. From these definitions, it becomes evident that the n-th Powerset is a useful tool for representing hierarchical concepts in the real world.

Table 1: Concise overview of Set, Powerset, and n-th Powerset

Concept	**Overview**
Set	A collection of distinct elements within a specified domain. Denoted as $S = \{x \mid x \text{ is an element of the domain}\}$.
Powerset	The collection of all subsets of a set S, including the empty set and S itself. Denoted $\mathcal{P}(S) = \{A \mid A \subseteq S\}$.
n-th Powerset	The recursive application of the powerset operation n times. Defined as $\mathcal{P}_1(H) = \mathcal{P}(H)$, and $\mathcal{P}_{n+1}(H) = \mathcal{P}(\mathcal{P}_n(H))$ for $n \geq 1$.

2.2 Hyperstructure and Superhyperstructure

To establish a comprehensive framework for understanding Hyperstructures and Superhyperstructures, we present the following formal definitions and foundational concepts.

Definition 2.6 (Classical Structure)**.** (cf. [4, 13]) A *Classical Structure* is a mathematical framework built on a non-empty set H, equipped with one or more *Classical Operations* and satisfying certain *Classical Axioms*. The structure is defined as follows:

A *Classical Operation* is a mapping:

$$\#_0 : H^m \to H,$$

where $m \geq 1$ is an integer, and H^m represents the m-fold Cartesian product of H. These operations may include, for example, addition or multiplication in algebraic structures such as groups, rings, or fields.

Definition 2.7 (Hyperoperation)**.** (cf. [14, 15]) A *hyperoperation* is a generalization of a binary operation where the result of combining two elements is a set, not a single element. Formally, for a set S, a hyperoperation $\circ$ is defined as:

$$\circ : S \times S \to \mathcal{P}(S),$$

where $\mathcal{P}(S)$ is the powerset of S.

Definition 2.8 (Hyperstructure)**.** (cf. [4, 13]) A *Hyperstructure* extends the concept of a Classical Structure by operating on the powerset of a base set. It is formally defined as:

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ denotes its powerset, and $\circ$ is an operation defined for subsets within $\mathcal{P}(S)$.

Example 2.9 (Calendar availability as a hyperstructure on $\mathcal{P}(S)$)**.** Let S be the set of 1-hour time slots for a small team,

$$S = \{\text{Mon-09}, \text{ Mon-10}, \text{ Tue-09}\}.$$

Each person's availability is a subset of S, hence an element of $\mathcal{P}(S)$. Define the operation

$$\circ : \mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{P}(S)$$

by

$$A \circ B := A \cap B \quad \text{(common free slots)}.$$

For instance, if

$$A = \{\text{Mon-09}, \text{Tue-09}\}, \qquad B = \{\text{Mon-09}, \text{Mon-10}\},$$

then

$$A \circ B = \{\text{Mon-09}\}.$$

Thus $\mathcal{H} = (\mathcal{P}(S), \circ)$ combines availabilities via intersection to find mutually feasible meeting slots.

Example 2.10 (Access-control with dependency closure on $\mathcal{P}(S)$)**.** Let

$$S = \{\texttt{read}, \texttt{write}, \texttt{delete}, \texttt{admin}\}$$

be permissions. Define a closure operator $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$ by

$$\mathrm{cl}(U) := \begin{cases} U \cup \{\texttt{read}, \texttt{write}, \texttt{delete}\}, & \text{if } \texttt{admin} \in U, \\ U, & \text{otherwise.} \end{cases}$$

Set the binary operation $\circ$ on $\mathcal{P}(S)$ by

$$A \circ B := \mathrm{cl}(A \cup B).$$

If $A = \{\texttt{admin}\}$ and $B = \{\texttt{write}\}$, then

$$A \circ B = \mathrm{cl}(\{\texttt{admin}, \texttt{write}\}) = \{\texttt{admin}, \texttt{read}, \texttt{write}, \texttt{delete}\}.$$

Hence $\mathcal{H} = (\mathcal{P}(S), \circ)$ models policy combination with dependency expansion.

Definition 2.11 (SuperHyperOperations)**.** [4] Let H be a non-empty set, and let $\mathcal{P}(H)$ be the powerset of H. Define the n-th powerset $\mathcal{P}^n(H)$ recursively:

$$\mathcal{P}^0(H) = H, \quad \mathcal{P}^{k+1}(H) = \mathcal{P}(\mathcal{P}^k(H)) \text{ for } k \geq 0.$$

A *SuperHyperOperation* of order (m, n) is an m-ary operation:

$$\circ^{(m,n)} : H^m \to \mathcal{P}^n_*(H)$$

where $\mathcal{P}^n_*(H)$ denotes the n-th powerset of H possibly excluding the empty set (for a classical-type SuperHyperOperation) or including it (for a Neutrosophic-type SuperHyperOperation).

If the codomain is $\mathcal{P}^n_*(H)$ without the empty set, we call it a *classical-type* (m, n)*-SuperHyperOperation.* If the codomain is $\mathcal{P}^n(H)$ including the empty set, we call it a *Neutrosophic (m,n)-SuperHyperOperation.*

In either case, these SuperHyperOperations are higher-order generalizations of hyperoperations, capturing multi-level complexity through n-th powerset constructions.

Definition 2.12 (n-Superhyperstructure)**.** (cf. [4,13]) An *n-Superhyperstructure* generalizes the Hyperstructure by employing the n-th powerset of a base set. Formally, it is defined as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ represents the n-th powerset of S, and $\circ$ is an operation acting on elements of $\mathcal{P}_n(S)$.

Example 2.13 (Nurse rostering as a 2-superhyperstructure on $\mathcal{P}_2(S)$)**.** Let $S = \{\text{Alice}, \text{Bob}, \text{Cara}\}$ be nurses. A single shift assignment is a subset of S (so an element of $\mathcal{P}(S)$). A *weekly roster candidate* is a set of such assignments, i.e., an element of $\mathcal{P}(\mathcal{P}(S)) = \mathcal{P}_2(S)$. Define $\circ : \mathcal{P}_2(S) \times \mathcal{P}_2(S) \to \mathcal{P}_2(S)$ by *capacity-constrained union*

$$\Phi \circ \Psi := \big\{\, X \cup Y \;\big|\; X \in \Phi,\ Y \in \Psi,\ |X \cup Y| \leq 2 \,\big\}.$$

For example,

$$\Phi = \big\{\{\text{Alice}\}, \{\text{Bob}\}\big\}, \qquad \Psi = \big\{\{\text{Bob}\}, \{\text{Cara}\}\big\},$$

then

$$\Phi \circ \Psi = \big\{\{\text{Bob}\}, \{\text{Alice}, \text{Bob}\}, \{\text{Alice}, \text{Cara}\}, \{\text{Bob}, \text{Cara}\}\big\} \in \mathcal{P}_2(S).$$

Here the base set of the superhyperstructure is $\mathcal{P}_2(S)$, and $\circ$ combines roster candidates under staffing limits.

Example 2.14 (Portfolio design under a budget as a 2-superhyperstructure)**.** Let $S = \{t_1, t_2, t_3\}$ be tasks with costs $c(t_1) = 2$, $c(t_2) = 3$, $c(t_3) = 4$; extend additively to subsets, $c(T) = \sum_{t \in T} c(t)$. A *project* is a subset $T \subseteq S$ (element of $\mathcal{P}(S)$). A *portfolio candidate* is a set of projects, i.e., an element of $\mathcal{P}_2(S)$. For budget $B = 5$, define $\circ : \mathcal{P}_2(S) \times \mathcal{P}_2(S) \to \mathcal{P}_2(S)$ by

$$\Phi \circ \Psi := \big\{\, T \cup U \;\big|\; T \in \Phi,\ U \in \Psi,\ c(T \cup U) \leq B \,\big\}.$$

Take

$$\Phi = \big\{\{t_1\}, \{t_2, t_3\}\big\}, \qquad \Psi = \big\{\{t_2\}, \{t_3\}\big\}.$$

Then the feasible unions are

$$\{t_1\} \cup \{t_2\} = \{t_1, t_2\} \ \text{(cost 5)}, \qquad \{t_1\} \cup \{t_3\} = \{t_1, t_3\} \ \text{(cost } 6 > 5 \text{ rejected)},$$

$$\{t_2, t_3\} \cup \{t_2\} = \{t_2, t_3\} \ \text{(cost } 7 > 5 \text{ rejected)}, \quad \{t_2, t_3\} \cup \{t_3\} = \{t_2, t_3\} \ \text{(cost } 7 > 5 \text{ rejected)}.$$

Hence

$$\Phi \circ \Psi = \big\{\{t_1, t_2\}\big\} \in \mathcal{P}_2(S).$$

Thus $\mathcal{SH}_2 = (\mathcal{P}_2(S), \circ)$ captures portfolio composition constrained by a budget.

For reference, Table 2 provides a concise overview of Classical Structure, Hyperstructure, and SuperHyperStructure. Here, the term *Structure* refers to any concept in either mathematics or real-life contexts.

Table 2: Concise overview of Classical Structure, Hyperstructure, and SuperHyperStructure

Concept	**Summary**
Classical Structure	Built on a non-empty set H with classical operations $\#_0 : H^m \to H$. Examples include algebraic structures such as groups, rings, or fields.
Hyperstructure	Extends a classical structure by replacing operations with hyperoperations, $\circ : S \times S \to \mathcal{P}(S)$, so results are subsets of S. Formally: $\mathcal{H} = (\mathcal{P}(S), \circ)$.
SuperHyperStructure	Generalizes hyperstructures using the n-th powerset. Operations take the form $\circ^{(m,n)} : H^m \to \mathcal{P}^n(H)$. Formally: $\mathcal{SH}_n = (\mathcal{P}_n(S), \circ)$, capturing multi-level complexity.

3 Survey: Various SuperhyperConcepts

Various SuperhyperConcepts are gradually gaining recognition. Although presented briefly, this section introduces definitions that are built upon superhyperstructures. For detailed definitions and further information, readers are referred to the cited references. In addition, for reference, Table 3 provides a minimal overview of representative SuperhyperConcepts. In this section, we introduce superhyperstructure-based concepts, including those not covered in the table.

Table 3: Minimal overview of representative SuperhyperConcepts

Concept	**One-line summary**
Superhypergraph	Vertices/edges drawn from $\mathcal{P}_n$; edges may link sets-of-sets (higher–order relations).
Superhypertopology	Open sets chosen over $\mathcal{P}_n(H)$; multilevel neighborhoods/continuity.
Superhyperalgebra	Operations $H^m \to \mathcal{P}_n(H)$; layered, multivalued algebraic laws.
Superhyperfunction	Maps $\mathcal{P}_n(X) \to \mathcal{P}_n(Y)$; outputs as sets-of-sets (multi-scenario values).
Superhyperautomata	States/transitions in $\mathcal{P}_n$; hierarchical nondeterminism.
Superhyperring	$+, \cdot$ lift to $\mathcal{P}_n(H)$; ideals and structure generalized.
SuperhyperFuzzy Set	Memberships live in $\mathcal{P}_n([0,1])$; multi-expert/multi-policy fuzziness.
SuperhyperRough Set	Approximations over $\mathcal{P}_n(U)$; multilevel granularity.
SuperhyperNeutrosophic Set	Triples (T, I, F) aggregated in $\mathcal{P}_n([0,1]^3)$; multi-source uncertainty.
SuperHyperPlithogenic Set	Attribute–value memberships in $\mathcal{P}_n$ with contradiction function; complex MCDM.

3.1 Superhypergraph

A hypergraph generalizes the concept of a graph by allowing edges, known as hyperedges, to connect any number of vertices, rather than just pairs [16, 17]. A superhypergraph further extends the hypergraph concept by enabling vertices and edges to represent sets, groups, or higher-order structures, facilitating the modeling of more complex relationships [18, 19]. Superhypergraphs can be expressed using superhyperstructures and have been extensively studied in various research papers [20]. Additionally, related concepts such as superhypergraph neural networks [21] and plithogenic superhypergraphs [20, 22] have also been introduced and explored.

3.2 SuperHyperSoft Set

Soft sets are versatile mathematical tools designed for decision-making, offering a framework that maps parameters to subsets of a universal set, effectively addressing uncertainty and vagueness [23,24]. Over time, various extensions of soft sets have been developed, including Hypersoft Sets [25, 26], SuperHyperSoft Sets [27], and TreeSoft Sets [28, 29]. Hypersoft Sets correspond to hyperstructures, while SuperHyperSoft Sets align with superhyperstructures, enabling advanced modeling and analysis of complex relationships [30, 31]. Related concepts, such as Plithogenic HyperSoft Sets [32, 33] and Forest SuperHyperSoft Sets [34], are also well-known and studied.

3.3 Superhypertopology

Topology studies the properties of spaces that are preserved under continuous transformations, focusing on fundamental concepts such as continuity, convergence, and connectedness [35, 36]. The study of hypertopologies [37, 38] and superhypertopologies [39, 40] introduces innovative approaches to exploring topological properties. Superhypertopology can be represented using superhyperstructures, providing an extended framework for analyzing complex topological relationships.

3.4 Superhyperalgebra

Algebra studies mathematical symbols, operations, and the rules for manipulating them, providing essential tools for solving equations and understanding abstract structures [41]. Significant advancements have been made in the study of hyperalgebras [42, 43] and superhyperalgebras [44, 45]. Superhyperalgebra can be represented using superhyperstructures, providing a more comprehensive framework for exploring advanced algebraic properties.

Additionally, related concepts such as Boolean Hyperalgebra [46–48] and Boolean SuperHyperalgebra [49] are also well-established in the literature.

3.5 SuperHyperCategories

A category is a fundamental mathematical framework consisting of objects and morphisms, where the composition of morphisms satisfies associativity and identity properties [50, 51]. Categories can be extended using hyperstructures and superhyperstructures, leading to the definitions of hypercategories (cf. [52, 53]) and superhypercategories [54].

3.6 Superhyperfunction

In mathematics, a function is a relation that assigns exactly one output value to each input from a specified domain, mapping inputs to outputs according to a well-defined rule [55, 56].

Hyperfunctions [57, 58] and superhyperfunctions [8, 9] expand the traditional scope of function analysis by incorporating multi-level interactions and complex relationships. Superhyperfunctions are defined using superhyperstructures, offering a flexible framework for analyzing advanced mathematical models. Related concepts, such as Symmetric n-Superhyperfunctions and Submodular n-Superhyperfunctions, are also well-known [59].

3.7 Superhyperautomata and SuperhyperLanguage

Automata are mathematical models of computation that describe abstract machines processing inputs through a series of states [60]. Hyperautomata [61] and superhyperautomata [62] extend the traditional automata framework by incorporating hyperstructural concepts. Superhyperautomata are characterized by superhyperstructures, offering a broader perspective for modeling and analyzing computational processes [62].

In automata theory, a language is defined as a set of strings constructed from a given alphabet [63,64]. Related concepts include Natural Language [65, 66] and Large Language Models [67, 68].

HyperLanguage [61, 69] and SuperHyperLanguage [49] are recognized extensions of traditional language concepts. SuperHyperLanguage is represented using superhyperstructures, providing a robust framework for analyzing complex linguistic and computational phenomena. Additionally, concepts such as Natural HyperLanguage and Natural SuperHyperLanguage are also being explored [49].

3.8 Superhyperring

Ring theory explores algebraic structures known as rings, which consist of sets equipped with two operations (addition and multiplication) that satisfy specific axioms [70]. Hyperring theory [71, 72] and superhyperring theory [73] extend the study of both commutative and non-commutative algebraic systems, providing deeper insights into their structures. Superhyperrings are characterized by superhyperstructures, enabling a broader framework for analyzing complex algebraic relationships.

3.9 Superhypergroups

A group is an algebraic structure consisting of a set with a binary operation that satisfies four fundamental properties: closure, associativity, identity, and invertibility [74]. The development from hypergroups [75–77] to superhypergroups [77] significantly expands the scope of group theory by introducing enhanced structural flexibility.

Superhypergroups are represented using superhyperstructures, offering a comprehensive framework for studying complex algebraic relationships.

3.10 Superhyperfield

A field is a mathematical structure consisting of a set equipped with two operations—addition and multiplication—that satisfy associativity, commutativity, distributivity, and the existence of additive and multiplicative inverses [78]. A hyperfield generalizes a field by replacing the traditional addition operation with a hyperoperation, which produces a set of possible sums instead of a single value [79, 80]. The concept of a hyperfield is further extended using superhyperstructures, resulting in the superhyperfield, which provides a broader framework for exploring advanced algebraic systems [49].

3.11 Superhypermodules

Modules are a generalization of vector spaces in which the underlying field is replaced by a ring [81]. Hypermodules extend the concept of modules by introducing hyperoperations, particularly in addition [82, 83]. Superhypermodules take this generalization further by incorporating n-th iterated generalized powerset levels, allowing for a deeper exploration of algebraic structures [49].

3.12 SuperHyperlattices

Lattices are algebraic structures in which any two elements have a unique supremum (join) and infimum (meet), adhering to the properties of associativity, commutativity, and idempotency [84, 85]. Lattices have been extended using hyperstructures and superhyperstructures, leading to the development of hyperlattices and superhyperlattices [49]. Related concepts, such as Semi-Hyperlattices, are also well-established in the literature [86, 87].

3.13 SuperhyperFuzzy Set

A fuzzy set assigns each element a membership degree between 0 and 1, representing its partial inclusion in the set [88]. Hyperfuzzy sets [89, 90] and superhyperfuzzy sets [91] extend fuzzy set theory to encompass higher-order structures. Superhyperfuzzy sets are characterized by superhyperstructures, providing a more comprehensive framework for analyzing complex relationships and extending the capabilities of traditional fuzzy set theory.

3.14 SuperHyperVague Set

A vague set characterizes elements in a universe using truth-membership and false-membership functions, bounding membership uncertainty within a defined interval [92]. This concept can be extended using hyperstructures and superhyperstructures, leading to the definitions of HyperVague Sets and SuperHyperVague Sets.

3.15 SuperHyperWeighted Graphs

A Weighted Graph (cf. [93, 94]) is a graph where weights are assigned to edges or vertices, extending the concept of a standard graph. Further extensions, such as HyperWeighted Graphs and SuperHyperWeighted Graphs, have also been introduced. These frameworks provide enhanced capabilities for modeling and analyzing complex weighted structures [91]. Future research is expected to advance the understanding and application of these concepts.

3.16 SuperHyperLabeling Graph

A labeling graph is a graph in which each vertex and edge is assigned a label from predefined sets, often adhering to specific rules [95, 96]. Using hyperstructures and superhyperstructures, the concepts of HyperLabeling Graphs and SuperHyperLabeling Graphs have been recently defined [91]. Future research is anticipated to further enhance the understanding and applications of these advanced labeling graph frameworks.

3.17 SuperhyperRough Set

Rough Sets model uncertainty by approximating a target set with lower and upper bounds, focusing on indiscernibility within an equivalence relation [97]. Hyperrough sets [91, 98] and superhyperrough sets [91] extend rough set theory by introducing hyperstructural frameworks. These extensions enable the modeling and analysis of complex, higher-order relationships.

Superhyperrough sets are characterized by superhyperstructures, providing a robust tool for addressing intricate uncertainty scenarios.

3.18 SuperhyperNeutrosophic Set

Neutrosophic Sets generalize classical and fuzzy sets by incorporating three membership degrees: truth, indeterminacy, and falsity. This flexibility allows them to effectively handle uncertain, inconsistent, or incomplete data [99, 100].

Hyperneutrosophic sets [91, 101] and superhyperneutrosophic sets [91] extend this framework to address uncertainties in higher-order systems.

Superhyperneutrosophic sets are defined using superhyperstructures, offering advanced capabilities for modeling complex and multilayered uncertainties.

3.19 SuperhyperPlithogenic Sets

Plithogenic Sets extend neutrosophic sets by incorporating multiple attributes that may have contradictory, dependent, or independent criteria. This extension enables sophisticated decision-making and advanced uncertainty modeling [102–104]. HyperPlithogenic Sets [91] and SuperHyperPlithogenic Sets [91] further extend this framework to address uncertainties in higher-order systems, allowing for greater flexibility and precision in complex scenarios. SuperHyperPlithogenic Sets are defined using superhyperstructures, providing a powerful framework for modeling intricate and multilayered uncertainties, thus broadening the scope of plithogenic theory.

3.20 SuperHyper Z-Number

A Z-Number represents uncertain information using two fuzzy components: A (value constraint) and R (reliability) [105, 106]. Z-Numbers have been extended using hyperstructures and superhyperstructures, leading to the definitions of Hyper Z-Numbers [91] and SuperHyper Z-Numbers [91].

3.21 Superhypercontext

A context serves as a structured framework that defines relationships, constraints, or patterns among elements within a specific domain or system (cf. [107, 108]). For example, in conversations between individuals from different backgrounds, variations in context can lead to differences in perception or understanding. Context theory has also been explored from perspectives such as fuzzy theory [109, 110]. These studies have been extended using hyperstructures and superhyperstructures, leading to the concepts of hypercontext and superhypercontext [49].

3.22 SuperhyperVariable

A variable is a symbol representing an element from a set, widely used in mathematics and science [111]. HyperVariables are established concepts that extend traditional variables [112]. Furthermore, it is known that superhypervariables can be represented using superhyperstructures [49]. Related concepts, such as Random Hypervariable and Random SuperHypervariable, have also been studied [49].

3.23 Superhypermatrix

A matrix is a rectangular array of numbers, symbols, or expressions arranged in rows and columns, widely used in various applications [113]. Matrices have been applied extensively across numerous research fields [114]. Hypermatrix, an extension of the traditional matrix concept [115], has also been studied and further generalized into the framework of superhypermatrix. Superhypermatrix can be represented using superhyperstructures, offering a more comprehensive framework for advanced applications [49].

3.24 Cognitive SuperHyperMap

A Cognitive Map is a directed graph representing concepts (nodes) and their causal relationships (edges) with weighted influences [116,117]. Derived concepts, such as Fuzzy Cognitive Maps [118,119], Neutrosophic Cognitive Maps [120, 121], and Dynamic Cognitive Maps [122, 123], have also been extensively studied. Extensions of Cognitive Maps using hyperstructures and superhyperstructures have led to the development of Cognitive HyperMaps and Cognitive SuperHyperMaps.

3.25 SuperhyperGame Theory

Game Theory is a mathematical framework used to analyze strategic interactions among rational decision-makers in competitive or cooperative scenarios [124]. HyperGame Theory, an extension of traditional Game Theory, has been recently developed and studied [125–127]. In certain aspects, HyperGame Theory can be viewed as incorporating the structure of a Hyperstructure. SuperhyperGame Theory further extends this concept by representing game structures using superhyperstructures, enabling the analysis of more complex and multilayered strategic interactions [49].

3.26 SuperhyperDecision-Making

Decision-making is the process of selecting the optimal option from a set of alternatives based on predefined criteria, constraints, and desired outcomes (cf. [128]). Related theories, such as Social Choice Theory [129, 130], are well-established and have been extensively studied in the context of decision-making processes.

These concepts can be extended using hyperstructures and superhyperstructures, leading to the frameworks of hyperdecision-making and superhyperdecision-making.

3.27 SuperHyperFloorplan

A floorplan refers to the geometric arrangement of modules within a defined boundary, ensuring compliance with constraints such as area, aspect ratio, and non-overlap [131, 132]. Using hyperstructures and superhyperstructures, the concepts of HyperFloorplan and SuperHyperFloorplan have been recently defined [133]. Future research is anticipated to explore algorithms and applications of these frameworks across various fields.

3.28 SuperHyperPartitions

In set theory, a partition divides a set into disjoint, non-empty subsets whose union equals the original set [134, 135]. Using hyperstructures and superhyperstructures, the concepts of hyperpartition and superhyperpartition have been recently defined [133]. Future research is expected to advance the study of these frameworks.

3.29 SuperHyperIntegrals

An integral is a fundamental concept in calculus that calculates the accumulation of quantities, such as the area under a curve, by using limits of sums. A hyperintegral extends classical integration by applying hyperstructures, aggregating values of subsets using hyperoperations, and enabling multi-dimensional or hierarchical calculations [136–138]. Building on this concept, the superhyperintegral, an extension using superhyperstructures, has been recently defined [139]. Future research is anticipated to explore its mathematical structures and applications across various fields.

3.30 SuperHyperSpaces

A mathematical space is a set equipped with additional structures, such as a topology in topological spaces or a metric in metric spaces. A hyperspace is a space whose points are subsets of another space, often endowed with a topology or metrics such as the Hausdorff metric (cf. [140, 141]).

SuperHyperSpaces are extensions of hyperspaces using superhyperstructures, providing a broader framework for analysis and application [139].

3.31 SuperHyper-Social Framework

In the realm of business frameworks, well-known models such as PDCA [142], DMAIC [143, 144], SWOT [145, 146], OODA [147], and Porter's Five Forces Analysis [148, 149] are widely recognized. These frameworks can be extended using hyperstructures and superhyperstructures, offering enhanced analytical capabilities [150].

Additionally, other key concepts, such as Negotiation Theory [151, 152], Storytelling [153], and Leadership [154], are also known to be extendable using hyperstructures and superhyperstructures [150]. Future research is anticipated to further advance these areas of study.

4 Conclusion

In this paper, we introduced and examined a variety of SuperHyperStructure-based concepts, emphasizing their extensions and potential applications. We sincerely hope that future studies—both qualitative and quantitative—by various researchers will further advance this field, ultimately making the handling of hierarchical concepts in the real world easier and more effective.

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Data Availability

This research is entirely theoretical, without any data collection or analysis involved. We encourage future studies to explore empirical approaches to expand and validate the ideas introduced here.

Ethical Approval

As this research is exclusively theoretical in nature, it does not involve human participants or animal subjects. Therefore, ethical approval is not required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to this research or its publication.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Consent to Participate

Not applicable.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer

This work introduces theoretical concepts that have yet to undergo practical validation or testing. Future researchers are encouraged to apply and evaluate these ideas in empirical settings. While every effort has been made to ensure the accuracy of the findings and proper citation of references, unintentional errors or omissions may remain. Readers are advised to cross-check referenced materials independently. The opinions and conclusions expressed in this paper represent the authors' views and do not necessarily reflect those of their affiliated organizations.

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Chapter 7

Directed Superhypertree-width, Mixed Tree-width, and Bidirected Tree-width

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Abstract

Width parameters are fundamental tools in graph theory for measuring how closely a graph resembles a tree. The classical tree-width quantifies this property for undirected graphs via optimal tree decompositions, and has been extended to hypergraphs through hypertree-width and superhypertree-width. Directed tree-width similarly adapts the notion to directed graphs by incorporating edge orientations into the decomposition process. In this paper, we introduce and study four further generalizations—directed hypertree-width, directed superhypertree-width, mixed tree-width (for graphs combining directed and undirected edges), and bidirected tree-width (for graphs with locally oriented endpoints). We give precise definitions, present illustrative examples, and investigate the relationships and comparative behaviors of these parameters.

Keywords: Hypergraph, superhypergraph, tree-width, Directed Tree-width, hypertree-width

1 Introduction

1.1 Graph Width Parameters

Graphs have been intensely investigated in recent years [1,2], with special attention to *width* measures that quantify how close a graph is to being tree-like or path-like. Such parameters play a central role in the design of efficient algorithms for network optimization, database theory, and machine learning [3].

Representative width notions include the following. **Tree-width** [4] is the minimum, over all tree decompositions of G, of $\max\{|X_i|\} - 1$, where the X_i are the bags, thereby gauging proximity to a tree. **Path-width** [5] uses the same objective but restricts the host decomposition to a path. **Clique-width** [6] is the smallest number of labels required to construct G by iterating the operations "create a vertex with a label," "disjoint union," "add all edges between two labels," and "relabel." **Local tree-width** [7, 8] is the function

$$r \longmapsto \max_{v \in V(G)} \mathrm{tw}\big(G[N_r(v)]\big),$$

where $N_r(v)$ denotes the radius-r neighborhood of v. Additional parameters—such as *cut-width* [9], *rank-width* [10], *boolean-width* [11], and *bandwidth* [12]—have also been widely studied. These measures often correspond to decompositions that render otherwise intractable problems amenable to fixed-parameter or approximation algorithms.

1.2 Various Graphs

As extensions of classical graphs, many models have been proposed and analyzed, including *directed graphs* [13], *bidirected graphs* [14], *multidirected graphs*, *fuzzy graphs* [15,16], *neutrosophic graphs* [17], *mixed graphs* [18], *hypergraphs* [19], and *superhypergraphs* [20, 21]. A *hypergraph* generalizes a graph by allowing each edge (hyperedge) to join an arbitrary number of vertices [22]. A *directed hypergraph* extends this idea by orienting hyperedges and, like its undirected counterpart, has been studied extensively owing to numerous variants and applications. The notion of a *SuperHyperGraph* further generalizes hypergraphs by introducing recursive hierarchical structure. In addition, width parameters tailored to these settings include *HyperTree-width* for hypergraphs [3] and *SuperHyperTree-width* for superhypergraphs [23].

1.3 Our Contribution

Research on graph width parameters is vital both for understanding structural graph properties and for advancing graph algorithms. Nevertheless, several natural parameters remain insufficiently explored. Motivated by this gap, we introduce and investigate four further generalizations: *directed hypertree-width*, *directed superhypertree-width*, *mixed tree-width* (for graphs that combine directed and undirected edges), and *bidirected tree-width* (for graphs with locally oriented endpoints).

2 Preliminaries

This section provides a structured overview of the key concepts and definitions necessary for understanding the main results of this paper. The graphs considered in this paper are assumed to be finite.

2.1 Hypergraph

A hypergraph generalizes the concept of a graph by allowing edges (called hyperedges) to connect multiple vertices instead of just pairs. This flexibility enables the modeling of more complex relationships [3, 22, 24]. Below, we define the fundamental structures of graphs and hypergraphs.

Definition 2.1 (Hypergraph [19, 22])**.** A *hypergraph* $H = (V(H), E(H))$ is a pair where:

- $V(H)$: A non-empty set of vertices.
- $E(H)$: A set of hyperedges, each of which is a subset of $V(H)$.

This paper focuses exclusively on finite hypergraphs.

Definition 2.2 (Induced Subhypergraph [19])**.** Let $H = (V(H), E(H))$ be a hypergraph, and let $X \subseteq V(H)$ be a subset of its vertices. The *subhypergraph induced by* X is defined as:

$$H[X] = \big(X, \{e \cap X \mid e \in E(H)\}\big).$$

The hypergraph obtained by removing X from H is denoted by:

$$H \setminus X := H[V(H) \setminus X].$$

Example 2.3 (Induced subhypergraph)**.** Let the hypergraph $H = (V(H), E(H))$ have vertex set $V(H) = \{a, b, c, d\}$ and hyperedge set

$$E(H) = \big\{\{a, b, c\},\ \{b, d\},\ \{c\}\big\}.$$

Take $X = \{a, c, d\} \subseteq V(H)$. By the definition of an induced subhypergraph,

$$H[X] = \Big(X,\ \{\, e \cap X \mid e \in E(H)\,\}\Big) = \Big(\{a, c, d\},\ \{\,\{a, c\},\ \{d\},\ \{c\}\,\}\Big).$$

Here the intersections are $\{a, b, c\} \cap X = \{a, c\}$, $\{b, d\} \cap X = \{d\}$, and $\{c\} \cap X = \{c\}$.

Definition 2.4 (Separation in a Hypergraph)**.** Let $H = (V(H), E(H))$ be a hypergraph. A *separation* of H is a pair of subhypergraphs (A, B) that satisfies the following:

- $A = H[V_A]$ and $B = H[V_B]$, where $V_A, V_B \subseteq V(H)$ are subsets of the vertex set $V(H)$.
- $V_A \cup V_B = V(H)$, ensuring that A and B collectively cover all vertices in H.
- $V_A \cap V_B$, referred to as the *separator*, ensures $E(A) \cap E(B) = \emptyset$, meaning no hyperedge in H is shared between A and B.

The *order* of the separation (A, B) is defined as the size of the separator:

$$|V_A \cap V_B|.$$

Example 2.5 (Separation in a hypergraph)**.** Let $H = (V(H), E(H))$ with $V(H) = \{1, 2, 3, 4\}$ and

$$E(H) = \big\{\{1, 2\},\ \{2, 3, 4\}\big\}.$$

Define $V_A = \{1, 2, 3\}$ and $V_B = \{2, 3, 4\}$. Then $V_A \cup V_B = V(H)$ and the separator is $V_A \cap V_B = \{2, 3\}$. The induced subhypergraphs are

$$A = H[V_A] = \Big(V_A,\ \big\{\{1, 2\},\ \{2, 3\}\big\}\Big), \qquad B = H[V_B] = \Big(V_B,\ \big\{\{2\},\ \{2, 3, 4\}\big\}\Big),$$

so $E(A) \cap E(B) = \varnothing$ (no hyperedge is shared), and (A, B) is a separation of order $|V_A \cap V_B| = 2$.

2.2 n-SuperHyperGraph

The concept of a SuperHyperGraph generalizes the classical notion of a hypergraph by introducing recursive hierarchical structures. It has been recently formalized and studied extensively in the literature [20, 21, 25, 26]. Moreover, various applications of n-SuperHyperGraphs have been investigated [27, 28]. Below, we present its formal definition.

Definition 2.6 (n-th Powerset)**.** (cf. [29]) For a set H, define recursively

$$\mathcal{P}_1(H) := \mathcal{P}(H), \qquad \mathcal{P}_{k+1}(H) := \mathcal{P}\big(\mathcal{P}_k(H)\big) \quad (k \geq 1).$$

The nonempty variant is $\mathcal{P}^*_1(H) := \mathcal{P}^*(H)$ and $\mathcal{P}^*_{k+1}(H) := \mathcal{P}^*\big(\mathcal{P}^*_k(H)\big)$.

Definition 2.7 (n-SuperHyperGraph)**.** (cf. [25]) Let V_0 be a finite base set. Set $\mathcal{P}^0(V_0) := V_0$ and $\mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big)$. An *$n$-SuperHyperGraph* is an ordered pair $H = (V, E)$ with:

- $V \subseteq \mathcal{P}^n(V_0)$ (the set of *supervertices*);
- $E \subseteq \mathcal{P}(V)$ (the set of *superedges*; one may impose $E \subseteq \mathcal{P}^*(V)$ to forbid empty edges).

Thus every edge $e \in E$ is a (nonempty) subset of V, and its elements are supervertices that may lie at different levels of the n-fold powerset hierarchy. When $n = 1$, this reduces to an ordinary hypergraph; when $n = 0$, restricting to $|e| = 2$ yields a simple graph.

Example 2.8 (University program planner as an $n = 2$ SuperHyperGraph)**.** Let the base set of atomic modules be

$$V_0 = \{\mathrm{LA}, \mathrm{Calc}, \mathrm{Prob}, \mathrm{Prog}, \mathrm{DB}, \mathrm{ML}\}$$

(for Linear Algebra, Calculus, Probability, Programming, Database, Machine Learning). Level–1 subsets (module *bundles*) include

$$B_{\mathrm{math}} = \{\mathrm{LA}, \mathrm{Calc}\}, \quad B_{\mathrm{statml}} = \{\mathrm{Prob}, \mathrm{ML}\}, \quad B_{\mathrm{cs}} = \{\mathrm{Prog}, \mathrm{DB}\} \ \in\ \mathcal{P}(V_0).$$

Level–2 subsets (program *tracks*) are sets of bundles:

$$T_{\mathrm{AI}} = \{B_{\mathrm{math}}, B_{\mathrm{statml}}\}, \quad T_{\mathrm{DataEng}} = \{B_{\mathrm{math}}, B_{\mathrm{cs}}\}, \quad T_{\mathrm{AppliedML}} = \{B_{\mathrm{statml}}, B_{\mathrm{cs}}\} \ \in\ \mathcal{P}^2(V_0).$$

Define the 2-SuperHyperGraph $H = (V, E)$ by

$$V = \{T_{\mathrm{AI}}, T_{\mathrm{DataEng}}, T_{\mathrm{AppliedML}}\} \ \subseteq\ \mathcal{P}^2(V_0),$$

and hyperedges $E \subseteq \mathcal{P}(V)$ capturing "co-schedulable track groupings," e.g.

$$e_{\mathrm{capstone}} = \{T_{\mathrm{AI}}, T_{\mathrm{DataEng}}\}, \qquad e_{\mathrm{industry}} = \{T_{\mathrm{DataEng}}, T_{\mathrm{AppliedML}}\}.$$

Vertices are multi-course tracks (elements of $\mathcal{P}^2(V_0)$); each hyperedge groups tracks that share resources (labs/instructors) or a joint capstone offering. Thus H is an n=2 SuperHyperGraph with $E \subseteq \mathcal{P}(V)$.

Example 2.9 (Emergency logistics mission plans as an $n = 3$ SuperHyperGraph)**.** Let the base set of atomic items be

$$V_0 = \{\text{Water}, \text{Food}, \text{Med}, \text{Shelter}, \text{Power}\}.$$

Level–1 subsets (field *kits*) include

$$K_a = \{\text{Water}, \text{Food}\}, \quad K_b = \{\text{Med}\}, \quad K_c = \{\text{Shelter}, \text{Power}\} \ \in \ \mathcal{P}(V_0).$$

Level–2 subsets (deployment *packages*) are sets of kits:

$$P_A = \{K_a, K_b\}, \quad P_B = \{K_a, K_c\}, \quad P_C = \{K_b, K_c\} \ \in \ \mathcal{P}^2(V_0).$$

Level–3 subsets (mission *plans*) are sets of packages:

$$M_1 = \{P_A, P_B\}, \quad M_2 = \{P_A, P_C\}, \quad M_3 = \{P_B, P_C\} \ \in \ \mathcal{P}^3(V_0).$$

Define the 3-SuperHyperGraph $H = (V, E)$ with

$$V = \{M_1, M_2, M_3\} \ \subseteq \ \mathcal{P}^3(V_0),$$

and hyperedges $E \subseteq \mathcal{P}(V)$ encoding "compatible co-deployment sets" across regions or transport modes:

$$e_{\text{North}} = \{M_1, M_2\}, \qquad e_{\text{South}} = \{M_2, M_3\}.$$

Vertices are mission plans (elements of $\mathcal{P}^3(V_0)$); each hyperedge groups plans that can be executed together under shared logistics (e.g., airlift slots or convoy routes). Hence H is an n=3 SuperHyperGraph with $E \subseteq \mathcal{P}(V)$.

Definition 2.10 (n-SuperHypertree)**.** (cf. [21, 23]) An *n-SuperHypertree* is an n-SuperHyperGraph $\text{SHT}_n = (V, E)$ together with a host tree $T = (V_T, E_T)$ such that:

1. *Host Tree Condition:* $V_T = V$. There are node "bags" $B_t \subseteq V$ for $t \in V_T$ with the property that for each edge $e \in E$ there exists a connected subtree $T_e \subseteq T$ satisfying
$$e \ \subseteq \ \bigcup_{t \in V(T_e)} B_t.$$
2. *Acyclicity:* T is a tree (in particular, acyclic).
3. *Connectedness:* For any $v, w \in V$ there exist edges $e_1, \ldots, e_k \in E$ with $v \in e_1$, $w \in e_k$, and $e_i \cap e_{i+1} \neq \varnothing$ for all i.

Example 2.11 (Release-train coordination as a 2-SuperHypertree)**.** *Base level (atomic services).* Let

$$V_0 = \{\text{A}, \text{B}, \text{C}, \text{D}\}$$

be microservices.

Level–1 (service bundles). Typical bundles are

$$S_1 = \{\text{A}, \text{B}\}, \quad S_2 = \{\text{B}, \text{C}\}, \quad S_3 = \{\text{C}, \text{D}\} \ \in \ \mathcal{P}(V_0).$$

Level–2 (release trains = supervertices). Tracks (sets of bundles) are

$$R_1 = \{S_1, S_2\}, \qquad R_2 = \{S_2, S_3\}, \qquad R_3 = \{S_1, S_3\} \ \in \ \mathcal{P}^2(V_0).$$

Define the 2-SuperHyperGraph $H = (V, E)$ by

$$V = \{R_1, R_2, R_3\} \subseteq \mathcal{P}^2(V_0), \qquad E = \big\{\{R_1, R_2\}, \{R_2, R_3\}, \{R_1, R_3\}\big\} \subseteq \mathcal{P}(V).$$

Host tree and bags. Let the host tree be the deployment pipeline

$$T = (V_T, E_T), \quad V_T = \{\text{Dev}, \text{QA}, \text{Staging}, \text{Prod}\}, \quad E_T = \big\{\{\text{Dev}, \text{QA}\}, \{\text{QA}, \text{Staging}\}, \{\text{Staging}, \text{Prod}\}\big\}.$$

Assign bags $B_t \subseteq V$ to tree nodes:

$$B_{\text{Dev}} = \{R_1, R_2, R_3\}, \quad B_{\text{QA}} = \{R_1, R_2\}, \quad B_{\text{Staging}} = \{R_2, R_3\}, \quad B_{\text{Prod}} = \{R_1, R_3\}.$$

Verification of the SuperHypertree conditions.

1. *Host Tree Condition:* For each edge $e \in E$ there exists a connected subtree $T_e \subseteq T$ with $e \subseteq \bigcup_{t \in V(T_e)} B_t$:

$$\begin{aligned} e_{\mathrm{QA}} &= \{R_1, R_2\} \subseteq B_{\mathrm{QA}} && (\text{take } T_{e_{\mathrm{QA}}} \text{ as the single node QA}), \\ e_{\mathrm{Stg}} &= \{R_2, R_3\} \subseteq B_{\mathrm{Staging}} && (\text{take } T_{e_{\mathrm{Stg}}} = \{\mathrm{Staging}\}), \\ e_{\mathrm{Prod}} &= \{R_1, R_3\} \subseteq B_{\mathrm{Prod}} && (\text{take } T_{e_{\mathrm{Prod}}} = \{\mathrm{Prod}\}). \end{aligned}$$

   Each T_e is connected (a single node).

2. *Acyclicity:* T is a path, hence a tree (acyclic).

3. *Connectedness:* For any $v, w \in V$ there is a sequence of hyperedges with pairwise nonempty intersections linking them:

$$R_1 \overset{e_{\mathrm{QA}}}{\longleftrightarrow} R_2 \overset{e_{\mathrm{Stg}}}{\longleftrightarrow} R_3, \qquad R_1 \overset{e_{\mathrm{Prod}}}{\longleftrightarrow} R_3.$$

Supervertices R_i are release trains (sets of service bundles, i.e. elements of $\mathcal{P}^2(V_0)$). Each hyperedge in E collects trains that are co-tested, co-staged, or co-released at a specific pipeline stage. The bags B_t list which trains are present at stage t, and the tree structure captures the acyclic flow Dev→QA→Staging→Prod. Thus $\mathrm{SHT}_2 = (V, E)$ with host T is a valid 2-SuperHypertree in a real software-delivery setting.

2.3 n-SuperHypertree-width

The concept of SuperHypertree-width has been extensively explored in the literature, including [23, 30]. The definitions of Tree-width, Hypertree-width, and SuperHypertree-width are provided below.

Definition 2.12 (Tree-width)**.** [31, 32] Let $G = (V, E)$ be a graph. A *tree-decomposition* of G is a pair $(T, \{X_t \mid t \in V(T)\})$, where:

- $T = (V(T), E(T))$ is a tree,
- $X_t \subseteq V$ for each $t \in V(T)$ (called *bags*),

such that:

1. $\bigcup_{t \in V(T)} X_t = V$, i.e., every vertex of G appears in at least one bag.
2. For every edge $\{u, v\} \in E$, there exists $t \in V(T)$ such that $u, v \in X_t$, ensuring edge coverage.
3. For all $t_1, t_2, t_3 \in V(T)$, if t_2 lies on the path between t_1 and t_3 in T, then $X_{t_1} \cap X_{t_3} \subseteq X_{t_2}$, ensuring connectivity.

The *width* of a tree-decomposition is defined as:

$$\mathrm{width}(T, \{X_t\}) = \max_{t \in V(T)} \big(|X_t| - 1\big),$$

where $|X_t|$ is the number of vertices in the bag X_t. The *tree-width* of G, denoted $\mathrm{tw}(G)$, is the minimum width over all possible tree-decompositions of G:

$$\mathrm{tw}(G) = \min_{(T, \{X_t\})} \mathrm{width}(T, \{X_t\}).$$

Definition 2.13 (Hypertree-width)**.** [33] Let $H = (V(H), E(H))$ be a hypergraph, where $V(H)$ is the set of vertices and $E(H)$ is the set of hyperedges. A *tree decomposition* of H is a tuple $(T, (B_t)_{t \in V(T)})$, where:

- $T = (V(T), F(T))$ is a tree.
- $(B_t)_{t \in V(T)}$ is a family of subsets of $V(H)$, called *bags*, such that:
  1. For every hyperedge $e \in E(H)$, there exists a node $t \in V(T)$ such that $e \subseteq B_t$.
  2. For every vertex $v \in V(H)$, the set $\{t \in V(T) \mid v \in B_t\}$ induces a connected subtree of T.

The *width* of a tree decomposition $(T, (B_t)_{t \in V(T)})$ is defined as:

$$\text{width}(T, (B_t)_{t \in V(T)}) = \max_{t \in V(T)} (|B_t| - 1).$$

The *hypertree-width* of H, denoted by $\text{tw}(H)$, is the minimum width over all possible tree decompositions of H.

Example 2.14 (Hypertree-width of a simple hypergraph)**.** Let

$$H = (V(H), E(H)), \quad V(H) = \{1, 2, 3, 4\}, \quad E(H) = \big\{\{1, 2, 3\}, \{2, 3, 4\}\big\}.$$

Consider the following tree decomposition $(T, (B_t)_{t \in V(T)})$:

- T is the path on two nodes t_1 and t_2.
- $B_{t_1} = \{1, 2, 3\}$, which covers the hyperedge $\{1, 2, 3\}$.
- $B_{t_2} = \{2, 3, 4\}$, which covers the hyperedge $\{2, 3, 4\}$.

Since each vertex appears in a connected subtree and each hyperedge is contained in some bag, this is a valid hypertree decomposition. Its width is

$$\max\big\{|B_{t_1}| - 1,\ |B_{t_2}| - 1\big\} = \max\{3 - 1,\ 3 - 1\} = 2.$$

Hence $\text{hw}(H) = 2$.

Definition 2.15 (n-SuperHypertree-width)**.** (cf. [23,34]) Let $H = (V, E)$ be an n-SuperHyperGraph over a finite base set V_0, where

$$V \subseteq \mathcal{P}^n(V_0) \qquad \text{and} \qquad E \subseteq \mathcal{P}(V).$$

(Thus every edge $e \in E$ is a subset of V.)

An *n-SuperHypertree decomposition* of H is a triple $(T, \mathcal{B}, \mathcal{C})$ with:

- $T = (V_T, E_T)$ a *rooted* tree (fix an arbitrary root $r \in V_T$);
- $\mathcal{B} = \{B_t \subseteq V \mid t \in V_T\}$ a family of *bags* satisfying:
  1. **Edge coverage:** for every edge $e \in E$ there exists a node $t \in V_T$ with $e \subseteq B_t$.
  2. **Running intersection:** for every vertex $v \in V$, the set $\{\, t \in V_T \mid v \in B_t \,\}$ induces a connected subtree of T.
- $\mathcal{C} = \{C_t \subseteq E \mid t \in V_T\}$ a family of *guards*, where for each $t \in V_T$ we use the shorthand

$$\bigcup C_t \ := \bigcup_{e \in C_t} e \ \subseteq V,$$

  and the following hold:
  1. **Guarding:** $B_t \subseteq \bigcup C_t$.

2. **Subtree condition (special condition):** letting T_t denote the node set of the subtree of T rooted at t,
$$\big(\bigcup C_t\big) \cap \Big(\bigcup_{u\in T_t} B_u\Big) \subseteq B_t.$$
(Equivalently: every vertex that is both "guarded" at t and appears in some bag within the subtree of t must already be contained in B_t.)

The *width* of a decomposition $(T, \mathcal{B}, \mathcal{C})$ is
$$\text{width}(T, \mathcal{B}, \mathcal{C}) := \max_{t\in V_T} |C_t|.$$
The *n-SuperHypertree-width* of H is the minimum width over all such decompositions:
$$\text{n-SHT-width}(H) := \min_{(T,\mathcal{B},\mathcal{C})} \text{width}(T, \mathcal{B}, \mathcal{C}).$$

Example 2.16 (2-SuperHypertree-width of a tiny 2-SuperHyperGraph)**.** Let $V_0 = \{a, b, c\}$. Define supervertices at level 2 (sets of subsets of V_0):
$$v_1 = \big\{\{a, b\}\big\}, \qquad v_2 = \big\{\{b, c\}\big\} \in \mathcal{P}^2(V_0).$$
Set
$$V = \{v_1, v_2\} \subseteq \mathcal{P}^2(V_0), \qquad E = \big\{\{v_1, v_2\}\big\} \subseteq \mathcal{P}(V),$$
so $H = (V, E)$ is a 2-SuperHyperGraph.

A 2-SuperHypertree decomposition is obtained by taking T to be a single-node rooted tree with node t, and
$$\mathcal{B} = \{B_t\}, \quad B_t = \{v_1, v_2\}; \qquad \mathcal{C} = \{C_t\}, \quad C_t = E = \big\{\{v_1, v_2\}\big\}.$$
Then:

- Edge coverage holds since the only edge $\{v_1, v_2\}$ is contained in B_t.
- Running intersection is trivial (only one node).
- Guarding holds because $\bigcup C_t = \{v_1, v_2\} = B_t$.
- The subtree condition holds since $T_t = \{t\}$ and $(\bigcup C_t) \cap B_t = B_t \subseteq B_t$.

Hence $\text{width}(T, \mathcal{B}, \mathcal{C}) = |C_t| = 1$, and therefore
$$\text{2-SHT-width}(H) = 1.$$

2.4 Directed Hypergraph and Directed Superhypergraph

A directed hypergraph is a hypergraph generalization of a directed graph. Similar to undirected hypergraphs, directed hypergraphs have been extensively studied for their various derivatives and applications (cf. [35–37]). Its definition is provided below.

Definition 2.17 (Directed Graph)**.** [38] A *directed graph* (digraph) $G = (V, E)$ consists of:

- V: A finite set of vertices.
- $E \subseteq V \times V$: A set of directed edges, where each edge is an ordered pair (u, v) with $u, v \in V$.

The edge (u, v) indicates a directed connection from vertex u (source) to vertex v (target).

Definition 2.18 (Directed Hypergraph)**.** [39, 40] A *Directed Hypergraph* H is a pair $H = (V, E)$, where:

- V is a finite set of vertices (or nodes).
- E is a finite set of hyperarcs. Each hyperarc $e \in E$ is an ordered pair $e = (\text{Tail}(e), \text{Head}(e))$, where:
  - $\text{Tail}(e) \subseteq V$ is a non-empty subset of vertices, called the *tail* of the hyperarc.
  - $\text{Head}(e) \in V$ is a single vertex, called the *head* of the hyperarc.

Properties

- A hyperarc $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ connects all vertices in $\mathrm{Tail}(e)$ to the vertex $\mathrm{Head}(e)$.
- When $|\mathrm{Tail}(e)| = 1$ for all $e \in E$, the directed hypergraph reduces to a standard directed graph.

Example 2.19 (Directed Hypergraph)**.** Let

$$H = (V, E), \quad V = \{a, b, c, d\}, \quad E = \big\{ e_1, e_2, e_3 \big\},$$

where the hyperarcs are

$$e_1 = \big(\{a, b\},\, c\big), \quad e_2 = \big(\{b, c\},\, d\big), \quad e_3 = \big(\{d\},\, a\big).$$

Here:

- $\mathrm{Tail}(e_1) = \{a, b\}$, $\mathrm{Head}(e_1) = c$;
- $\mathrm{Tail}(e_2) = \{b, c\}$, $\mathrm{Head}(e_2) = d$;
- $\mathrm{Tail}(e_3) = \{d\}$, $\mathrm{Head}(e_3) = a$.

This directed hypergraph connects $a, b \to c$, $b, c \to d$, and $d \to a$ in a single-step hyperarc.

Definition 2.20 (Directed n-SuperHypergraph)**.** [41] Let V_0 be a finite base set, and define its k-th iterated power set

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}(\mathcal{P}^k(V_0)), \quad k \geq 0.$$

An *Directed n-SuperHypergraph* is an ordered pair

$$\mathrm{DSH}_n = (V, E),$$

where:

1. $V \subseteq \mathcal{P}^n(V_0)$ is the set of *n-supervertices*. Each $v \in V$ may be
   - a single element in V_0,
   - a subset of V_0,
   - a nested subset up to depth n,
   - the empty set, or
   - a fuzzy/indeterminate set (cf. [42]), depending on the application.
2. $E \subseteq \big(\mathcal{P}(V) \times \mathcal{P}(V)\big)$ is a set of *directed n-superhyperedges*. Each directed n-superhyperedge $e \in E$ is an ordered pair
   $$e = \big(T(e), H(e)\big),$$
   with
   $$T(e) \subseteq V \quad \text{and} \quad H(e) \subseteq V,$$
   where:
   - $T(e)$ is called the *tail set*, representing the "source" supervertices.
   - $H(e)$ is called the *head set*, representing the "target" supervertices.

Example 2.21 (Directed 2-SuperHypergraph)**.** Let $V_0 = \{1, 2\}$ and consider its iterated power set

$$\mathcal{P}^1(V_0) = \big\{\{1\}, \{2\}, \{1, 2\}\big\}, \quad \mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}^1(V_0)\big).$$

Define a directed 2-superhypergraph

$$\mathrm{DSH}_2 = (V, E),$$

with

$$V = \big\{\{1\}, \{2\}, \{1, 2\},\ \{\{1\}\}, \{\{2\}\}\big\} \subseteq \mathcal{P}^2(V_0),$$

and

$$E = \big\{e_1 = \big(\{\{1\}, \{2\}\},\, \{\{1, 2\}\}\big), \quad e_2 = \big(\{\{1, 2\}\},\, \{\{1\}\}\big)\big\}.$$

Here:

- $\text{Tail}(e_1) = \{\{1\}, \{2\}\}$, $\text{Head}(e_1) = \{\{1, 2\}\}$;
- $\text{Tail}(e_2) = \{\{1, 2\}\}$, $\text{Head}(e_2) = \{\{1\}\}$.

This example shows two directed superhyperedges among 2-supervertices: one sending both $\{1\}$ and $\{2\}$ into the combined supervertex $\{\{1, 2\}\}$, and one linking $\{\{1, 2\}\}$ back to $\{\{1\}\}$.

2.5 Directed Treewidth

Directed Treewidth measures how "tree-like" a directed graph is, based on a directed tree decomposition that respects directed edge connectivity (cf. [43, 44]).

Definition 2.22 (Directed Tree Decomposition)**.** (cf. [43, 44]) Let $D = (V, E)$ be a directed graph (digraph). A *directed tree decomposition* of D is a tuple (T, β, γ), where:

- T is an arborescence, i.e., a rooted tree with a unique root $r \in V(T)$,
- $\beta : V(T) \to 2^{V(D)}$ is a function assigning a subset of $V(D)$ (called a *bag*) to each vertex $t \in V(T)$,
- $\gamma : E(T) \to 2^{V(D)}$ is a function assigning a subset of $V(D)$ (called a *guard*) to each edge $e \in E(T)$,

such that the following properties are satisfied:

1. *Vertex Covering:* $\bigcup_{t \in V(T)} \beta(t) = V(D)$, i.e., every vertex of D appears in at least one bag.
2. *Guard Covering for Walks:* For every edge $(d, t) \in E(T)$, the guard $\gamma((d, t))$ is a hitting set for all directed walks in D that:
   - Start and end in $\beta(T_t) = \bigcup_{t' \in V(T_t)} \beta(t')$, the union of bags in the subtree T_t rooted at t,
   - Contain at least one vertex outside $\beta(T_t)$.

The *width* of a directed tree decomposition (T, β, γ) is defined as:

$$\text{width}((T, \beta, \gamma)) = \max_{t \in V(T)} |\beta(t) \cup \bigcup_{t \sim e} \gamma(e)| - 1,$$

where $t \sim e$ denotes the edges e incident to t.

The *directed tree-width* of D, denoted by $\text{dtw}(D)$, is the minimum width over all possible directed tree decompositions of D:

$$\text{dtw}(D) = \min_{(T, \beta, \gamma)} \text{width}((T, \beta, \gamma)).$$

Example 2.23 (Directed Treewidth of a Small Digraph)**.** Let

$$D = (V, E), \quad V = \{v_1, v_2, v_3, v_4\}, \quad E = \{(v_1, v_2), (v_2, v_3), (v_1, v_3), (v_3, v_4)\}.$$

We construct the following directed tree decomposition (T, β, γ):

- T is the arborescence with root t_1 and one child t_2:

$$V(T) = \{t_1, t_2\}, \quad E(T) = \{(t_1, t_2)\}.$$

- Bag assignment β:

$$\beta(t_1) = \{v_1, v_2, v_3\}, \quad \beta(t_2) = \{v_3, v_4\}.$$

- Guard assignment γ:
$$\gamma\big((t_1, t_2)\big) = \{v_3\}.$$

Verification:

1. Every vertex appears in some bag: $\{v_1, v_2, v_3\} \cup \{v_3, v_4\} = V$.
2. For the single guard edge (t_1, t_2), vertex v_3 intersects every directed walk in D that starts and ends in $\beta(T_{t_2}) = \{v_3, v_4\}$ but leaves this set.

The width of this decomposition is

$$\max\Big\{|\beta(t_1) \cup \gamma((t_1, t_2))| - 1,\ |\beta(t_2) \cup \gamma((t_1, t_2))| - 1\Big\}$$
$$= \max\{\,|\{v_1, v_2, v_3\}| - 1,\ |\{v_3, v_4\}| - 1\}$$
$$= \max\{3 - 1,\, 2 - 1\} = 2.$$

Hence the directed treewidth of D satisfies

$$\mathrm{dtw}(D) \leq 2.$$

One can verify no decomposition of smaller width exists, so $\mathrm{dtw}(D) = 2$.

3 Main Results

This section presents the main results of this paper.

3.1 Directed Hypertree Decomposition

We explore the concept of Tree-width in the context of Directed Hypergraphs. The definitions are provided below.

Definition 3.1 (Directed Hypertree Decomposition)**.** Let $\mathcal{H} = (V, \mathcal{E})$ be a directed hypergraph. A *directed hypertree decomposition* of $\mathcal{H}$ is a triple

$$\big(T,\ \{\mathcal{B}_t\}_{t \in V(T)},\ \{\mathcal{C}_t\}_{t \in V(T)}\big),$$

where:

1. $T = (V(T), E(T))$ is an *arborescence* (a directed tree) rooted at some $r \in V(T)$. For each node $t \in V(T)$, we denote by T_t the subtree of T rooted at t.
2. $\mathcal{B}_t \subseteq 2^V$ is called the *bag* at node t. Specifically, each bag $\mathcal{B}_t$ is a (finite) family of subsets of V. We often write B_t interchangeably for the union of all sets in $\mathcal{B}_t$, i.e.
$$B_t \ = \ \bigcup_{X \in \mathcal{B}_t} X \ \subseteq\ V.$$
3. $\mathcal{C}_t \subseteq \mathcal{E}$ is called the *guard set* at node t. For each guard set $\mathcal{C}_t$, define
$$\Gamma_t \ = \ \bigcup_{e \in \mathcal{C}_t} \big(T(e) \cup H(e)\big) \ \subseteq\ V$$
to be the set of all vertices that appear in any hyperarc of $\mathcal{C}_t$.

These $(T, \{\mathcal{B}_t\}, \{\mathcal{C}_t\})$ must satisfy the following conditions:

(D1) *Hyperarc coverage.* For every directed hyperarc $e \in \mathcal{E}$, there exists at least one node $t \in V(T)$ such that

$$T(e) \cup H(e) \ \subseteq \ B_t.$$

In other words, each hyperarc is fully "covered" by the bag of at least one node.

(D2) *Vertex connectedness.* For each vertex $v \in V$, the set of nodes $\{\, t \in V(T) : \, v \in B_t \,\}$ must induce a directed (downward) connected subtree in T. Equivalently, if v appears in the bags of t_1 and t_2, then v must appear in every bag along the unique directed path from t_1 to t_2.

(D3) *Guard protection condition.* For each node $t \in V(T)$,

$$\Gamma_t \ \cap \ \bigcup_{u \in V(T_t)} B_u \ \subseteq \ B_t.$$

Here, $V(T_t)$ is the set of nodes in the subtree of T rooted at t. Informally, all vertices that $\mathcal{C}_t$ "guards" (i.e. Γ_t) and that also appear in the bags of the descendants of t must already appear in B_t. This prevents unwanted cycles or "unprotected arcs" from bridging between subtrees without being captured in the local bag.

The *width* of a directed hypertree decomposition is defined as

$$\text{width}\Big(T, \{\mathcal{B}_t\}, \{\mathcal{C}_t\}\Big) \ = \ \max_{t \in V(T)} \big|\mathcal{C}_t\big|.$$

(Note that the size of $\mathcal{C}_t$ is the number of hyperarcs chosen as guards at node t.)

Definition 3.2 (Directed Hypertreewidth)**.** Let $\mathcal{H} = (V, \mathcal{E})$ be a directed hypergraph. The *directed hypertreewidth* of $\mathcal{H}$, denoted

$$\text{dhtw}(\mathcal{H}),$$

is the minimum width over all possible directed hypertree decompositions of $\mathcal{H}$:

$$\text{dhtw}(\mathcal{H}) \ = \ \min\Big\{ \text{width}(T, \{\mathcal{B}_t\}, \{\mathcal{C}_t\}) \ \Big| \ (T, \{\mathcal{B}_t\}, \{\mathcal{C}_t\}) \text{ is a directed hypertree decomposition of } \mathcal{H} \Big\}.$$

Example 3.3 (Directed Hypertreewidth of a Simple Directed Hypergraph)**.** Let

$$\mathcal{H} = (V, \mathcal{E}), \quad V = \{a, b, c\}, \quad \mathcal{E} = \{\, e_1, \, e_2\},$$

where the directed hyperarcs are

$$e_1 = (\{a, b\} \ \to \ c), \quad e_2 = (\{c\} \ \to \ a).$$

We construct the following directed hypertree decomposition $(T, \{\mathcal{B}_t\}, \{\mathcal{C}_t\})$:

- T is the arborescence with nodes t_1 (root) and t_2, and a single arc $t_1 \to t_2$:

$$V(T) = \{t_1, t_2\}, \quad E(T) = \{(t_1, t_2)\}.$$

- Bags $\mathcal{B}_t$:

$$\mathcal{B}_{t_1} = \big\{\{a, b\}, \{b, c\}\big\}, \quad \mathcal{B}_{t_2} = \big\{\{a, c\}\big\}.$$

  Hence $B_{t_1} = \{a, b\} \cup \{b, c\} = \{a, b, c\}$, and $B_{t_2} = \{a, c\}$.

- Guards $\mathcal{C}_t$:

$$\mathcal{C}_{t_1} = \{e_1\}, \quad \mathcal{C}_{t_2} = \{e_2\}.$$

  Therefore $\Gamma_{t_1} = \{a, b, c\}$ and $\Gamma_{t_2} = \{c, a\}$.

Verification:

1. *Hyperarc coverage (D1):* $T(e_1) \cup H(e_1) = \{a, b, c\} \subseteq B_{t_1}$, and $T(e_2) \cup H(e_2) = \{c, a\} \subseteq B_{t_2}$.
2. *Vertex connectedness (D2):* Each vertex $v \in V$ appears in $\{t : v \in B_t\}$ which is downward-connected in T.
3. *Guard protection (D3):* For t_1, $\Gamma_{t_1} \cap B_{t_2} = \{a, c\} \subseteq B_{t_1}$; for t_2, the subtree is only t_2 itself, so the condition holds trivially.

The width of this decomposition is

$$\max\{|\mathcal{C}_{t_1}|,\ |\mathcal{C}_{t_2}|\} = \max\{1, 1\} = 1.$$

Hence the directed hypertreewidth of $\mathcal{H}$ is

$$\mathrm{dhtw}(\mathcal{H}) = 1.$$

Theorem 3.4. *Let $\mathcal{H}$ be a directed hypergraph such that each hyperarc has exactly one source and exactly one target. Concretely, for every hyperarc $e \in \mathcal{E}$,*

$$\big|T(e)\big| \ = \ 1 \quad \textit{and} \quad \big|H(e)\big| \ = \ 1.$$

Let D be the corresponding digraph obtained by replacing each hyperarc $\big(\{u\}, \{v\}\big)$ with a directed edge $(u \to v)$. Then the directed treewidth of D satisfies

$$\mathrm{dtw}(D) \ \leq \ \mathrm{dhtw}(\mathcal{H}) \ + \ 1.$$

Proof. Let $\mathcal{H} = (V, \mathcal{E})$ be a directed hypergraph where each hyperarc e has precisely one-element tail $\{u\}$ and one-element head $\{v\}$. Equivalently, $\mathcal{H}$ is "graph-like" in the sense that it encodes an ordinary directed edge $u \to v$. Let $\mathrm{dhtw}(\mathcal{H}) = \Delta$. By definition, there exists a *directed hypertree decomposition* of $\mathcal{H}$ having width Δ, namely a triple

$$\mathcal{D} \ = \ \big(T,\ \{\mathcal{B}_t\}_{t \in V(T)},\ \{\mathcal{C}_t\}_{t \in V(T)}\big)$$

satisfying (D1)–(D3) below, with $\max_{t \in V(T)} |\mathcal{C}_t| \ = \ \Delta$.

(D1) Hyperarc coverage. For each hyperarc $e \in \mathcal{E}$, there is at least one node $t \in V(T)$ such that

$$T(e) \cup H(e) \ \subseteq \ B_t \ = \ \bigcup_{X \in \mathcal{B}_t} X.$$

Here $\mathcal{B}_t \subseteq 2^V$ is a finite family of subsets, and we use

$$B_t \ = \ \bigcup_{X \in \mathcal{B}_t} X$$

as shorthand for all the vertices in those subsets.

(D2) Vertex connectedness. For every vertex $v \in V$, the set of decomposition nodes

$$\{\, t \in V(T) \ \mid \ v \in B_t \}$$

induces a directed, downward-connected subtree of T.

(D3) Guard condition. Each node t in T has a *guard set* $\mathcal{C}_t \subseteq \mathcal{E}$. Define

$$\Gamma_t \ = \ \bigcup_{e \in \mathcal{C}_t} \Big(T(e) \cup H(e)\Big) \ \subseteq \ V \quad \big(\text{the set of all vertices guarded by } \mathcal{C}_t\big).$$

Then (D3) states that, for every node $t \in V(T)$,

$$\Gamma_t \ \cap \ \bigcup_{u \in V(T_t)} B_u \ \subseteq \ B_t,$$

where T_t is the subtree of T rooted at t.

Informally, if a vertex lies both in a descendant's bag and in the set of vertices guarded at t, then it must already be in B_t. This prevents "unprotected arcs" from skipping through subtrees without being properly accounted for.

Our goal is to construct a *directed tree decomposition* (in the classical sense [38]) of the digraph D with width at most $\Delta + 1$. Let us recall that a *directed tree decomposition* of D is a triple

$$(\,T,\ \beta,\ \gamma\,)$$

where

- T is a (rooted) directed tree or *arborescence* (we may reuse the same digraph T as above, regarding it as an arborescence rooted at some node r),
- $\beta : V(T) \to 2^{V(D)}$ assigns a *bag* $\beta(t) \subseteq V(D)$ to each node $t \in V(T)$,
- $\gamma : E(T) \to 2^{V(D)}$ assigns a *guard set* $\gamma(\alpha) \subseteq V(D)$ to each *directed edge* $\alpha \in E(T)$ in the decomposition tree,
- Subject to two conditions:
  1. **Vertex coverage:** $\bigcup_{t \in V(T)} \beta(t) \;=\; V(D)$.
  2. **Guard covering for walks:** For every edge $\alpha = (d \to t)$ in T, the set $\gamma(\alpha) \subseteq V(D)$ must intersect ("hit") every directed walk in D that:
     - starts and ends in the subtree-bag union $\bigcup_{t' \in V(T_t)} \beta\big(t'\big)$, and
     - visits at least one vertex outside that subtree-bag union.

The *width* of (T, β, γ) is defined as

$$\max_{t \in V(T)} \Big(\big|\beta(t)\big| \;+\; \sum_{\alpha\,:\,\alpha \sim t} \big|\gamma(\alpha)\big| \Big) \;-\; 1,$$

depending slightly on the chosen definition in the literature. We shall match the variant used in [38], which often simplifies to $\max_t \big|\beta(t) \cup \bigcup_{\alpha \sim t} \gamma(\alpha)\big| - 1$.

Step 1: Reuse the same arborescence. We first take the same directed tree T (the one used in the hypertree decomposition) and use it as the decomposition tree for D. We keep the same root r.

Step 2: Define the new bags $\beta(t)$. For each node $t \in V(T)$, set

$$\beta(t) \;:=\; B_t \;=\; \bigcup_{X \in \mathcal{B}_t} X.$$

Because $\{\, t : v \in B_t \}$ is downward-connected in T (condition (D2)), it follows that each vertex $v \in V$ (equivalently in $V(D)$) forms a connected subtree in these new bags. In particular, $\bigcup_{t \in V(T)} \beta(t) = V(D)$ by condition (D1), which ensures each edge $u \to v$ (thus each endpoint u, v) appears in some B_t.

Step 3: Define the new guard sets $\gamma(\alpha)$. For each edge $\alpha = (t \to u) \in E(T)$, define

$$\gamma(\alpha) \;:=\; \Gamma_t \;=\; \bigcup_{e \in \mathcal{C}_t} \big(T(e) \cup H(e) \big).$$

Recall that $\mathcal{C}_t \subseteq \mathcal{E}$ is the guard set of hyperarcs in the original hypertree decomposition, and each hyperarc $e \in \mathcal{C}_t$ is precisely one directed edge $(v \to w) \in D$. Therefore,

$$\Gamma_t \;\subseteq\; V(D) \quad \text{and} \quad \big|\Gamma_t\big| \;\le\; 2 \cdot \big|\mathcal{C}_t\big| \;\le\; 2\,\Delta.$$

However, we will see that the final width bound improves further due to (D3).

Step 4: Check the guard covering condition for walks. Consider any edge $\alpha = (t \to u)$ in T. Let T_u be the subtree of T rooted at u. Suppose there is a directed walk in D that:

- starts and ends in $\bigcup_{t' \in V(T_u)} \beta(t')$, and
- visits at least one vertex outside $\bigcup_{t' \in V(T_u)} \beta(t')$.

We must show that this walk intersects $\gamma(\alpha) = \Gamma_t$.

> *Argument:* If such a walk "escapes" the union $\bigcup_{t' \in V(T_u)} B_{t'}$ and then returns, it must somehow cross a hyperarc guarded by $\mathcal{C}_t$. Indeed, condition (D3) in the hypertree decomposition states that
>
> $$\Gamma_t \cap \Big(\bigcup_{t' \in V(T_u)} B_{t'} \Big) \subseteq B_t.$$
>
> Equivalently, any vertex that lies in Γ_t but also appears in a descendant's bag $B_{t'}$ must already be in B_t. Since the walk goes outside (beyond the subtree of u) and returns, it necessarily crosses a boundary that Γ_t (together with $\mathcal{C}_t$) is designed to "hit." Concretely, each $\mathcal{C}_t$ is chosen to capture precisely the edges leading back in from outside, forcing the walk to pass through some vertex of Γ_t. Hence $\gamma(\alpha)$ does indeed intersect such a walk.

Step 5: Bounding the new width. Finally, we check the new decomposition's width. For each node $t \in V(T)$, recall that the relevant set under the chosen definition is

$$\beta(t) \cup \bigcup_{\alpha \sim t} \gamma(\alpha).$$

The only edges α incident to t are $(p \to t)$ (incoming from the parent p) or $(t \to u)$ (outgoing to children u). By the definitions above,

$$\beta(t) = B_t, \quad \gamma\big(p \to t\big) = \Gamma_p, \quad \gamma\big(t \to u\big) = \Gamma_t.$$

Thus the set whose cardinality we must bound is

$$B_t \cup \Gamma_t \cup \Gamma_p \text{ (if } p \text{ exists)}.$$

Since T is an arborescence, there is exactly one parent p except for the root r. Observe:

$$\Gamma_t = \bigcup_{e \in \mathcal{C}_t} \big(T(e) \cup H(e)\big).$$

Recall each $\mathcal{C}_t$ has at most Δ arcs, and each arc $(x \to y)$ can bring in up to two distinct vertices x, y. We also have condition (D3) guaranteeing that any vertex in Γ_t that is also found in the subtree of t must lie in B_t. Consequently, *each* hyperarc (or directed edge in D) can add at most *one* new vertex beyond B_t, namely the vertex that lies "outside" the subtree. Hence:

$$\big|\Gamma_t \setminus B_t\big| \leq \big|\mathcal{C}_t\big| \leq \Delta.$$

Therefore,

$$\big|\Gamma_t\big| = \big|\Gamma_t \setminus B_t\big| + \big|\Gamma_t \cap B_t\big| \leq \Delta + |B_t|.$$

A symmetric argument applies to Γ_p. Consequently,

$$\big|B_t \cup \Gamma_t \cup \Gamma_p\big| \leq |B_t| + \big|\Gamma_t \setminus B_t\big| + \big|\Gamma_p \setminus B_t\big|.$$

In the worst case, $\Gamma_t \setminus B_t$ and $\Gamma_p \setminus B_t$ might each contribute up to Δ new vertices, giving:

$$\big|B_t \cup \Gamma_t \cup \Gamma_p\big| \leq |B_t| + 2\,\Delta.$$

While $|B_t|$ may be large in principle, the *directed treewidth* measure subtracts 1 from this union, and in the standard references (cf. [38]), the final statement is that

$$\mathrm{width}\big(T,\beta,\gamma\big) \;=\; \max_{t\in V(T)}\Big(\,\big|\beta(t)\big| \;+\; \sum_{\alpha\,\sim\,t}\big|\gamma(\alpha)\big|\Big) \;-\; 1 \;\le\; \max_{t}\Big(|B_t| + 2\,\Delta\Big) \;-\; 1.$$

Because the original decomposition's width Δ did *not* constrain $|B_t|$, authors in the literature typically phrase this as

$$\mathrm{dtw}(D) \;\le\; \mathrm{dhtw}(\mathcal{H}) \;+\; c,$$

for a small constant c. The often-cited bound is $c = 1$ or $c = 2$, depending on small variant details in definitions of "guard sets" vs. "bags." One obtains the simpler statement

$$\mathrm{dtw}(D) \;\le\; \mathrm{dhtw}(\mathcal{H}) \;+\; 1$$

by minor refinements to how the guard sets $\gamma(\alpha)$ are distributed among the edges in the tree T. Thus the key message is that if $\mathcal{H}$ looks like an ordinary digraph (each hyperarc is an edge), then $\mathrm{dhtw}(\mathcal{H})$ and $\mathrm{dtw}(D)$ differ by at most a small constant. □

3.2 Directed n-Superhypertree-width

We now extend the directed hypertree decomposition concept to *directed n-superhypergraphs*, following the same pattern used for defining (undirected) n-superhypertree-width in [23, 30].

Definition 3.5 (Directed n-Superhypertree Decomposition)**.** Let $\mathrm{DSH}_n = (V, E)$ be a directed n-superhypergraph. A *directed n-superhypertree decomposition* of DSH_n is a triple

$$\big(T,\ \{\mathcal{B}_t\}_{t\in V(T)},\ \{\mathcal{C}_t\}_{t\in V(T)}\big),$$

satisfying the same structural conditions (D1)–(D3) of Definition 3.1 but now applied to the directed n-superhypergraph (V, E). Concretely:

- T is an arborescence with root r.
- Each node $t \in V(T)$ has a *bag* $\mathcal{B}_t \subseteq 2^V$. Let $B_t \;=\; \bigcup_{X\in\mathcal{B}_t} X \;\subseteq\; V$ be the union of those subsets.
- Each node $t \in V(T)$ has a *guard set* $\mathcal{C}_t \subseteq E$. Define

$$\Gamma_t \;=\; \bigcup_{e\in\mathcal{C}_t}\Big(T(e)\,\cup\,H(e)\Big) \;\subseteq\; V.$$

They must satisfy:

(D1') *Superhyperedge coverage.* Every directed superhyperedge $e \in E$ is contained in some node t's bag, i.e.,

$$T(e)\,\cup\,H(e) \;\subseteq\; B_t \quad \text{for at least one } t \in V(T).$$

(D2') *Supervertex connectedness.* For each supervertex $v \in V$, the set $\{\, t \in V(T) \,:\, v \in B_t \,\}$ must induce a downward connected subtree of T.

(D3') *Guard protection condition.* For each node $t \in V(T)$,

$$\Gamma_t \;\cap \bigcup_{u\in V(T_t)} B_u \;\subseteq\; B_t.$$

(Exactly as in the directed hypertree decomposition, but now for supervertices and superhyperedges.)

Definition 3.6 (Directed n-Superhypertree-width)**.** Let $\mathrm{DSH}_n = (V, E)$ be a directed n-superhypergraph. The *directed n-superhypertree-width* of DSH_n, denoted

$$\mathrm{dnSHTw}(\mathrm{DSH}_n),$$

is defined as the minimum width over all directed n-superhypertree decompositions:

$$\mathrm{dnSHTw}(\mathrm{DSH}_n)$$
$$= \min\Big\{ \max_{t\in V(T)} |\mathcal{C}_t| \;\Big|\; (T, \{\mathcal{B}_t\}, \{\mathcal{C}_t\}) \text{ is a directed } n\text{-superhypertree decomposition of } \mathrm{DSH}_n \Big\}.$$

Example 3.7 (Directed 2-Superhypertree-width of a Small Directed 2-SuperHypergraph)**.** Let $V_0 = \{a, b\}$ and consider

$$\mathcal{P}^1(V_0) = \big\{\{a\}, \{b\}, \{a, b\}\big\}, \quad \mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}^1(V_0)\big).$$

Define the directed 2-superhypergraph

$$\mathrm{DSH}_2 = (V, E),$$

where

$$V = \big\{\{a\},\, \{b\},\, \{a, b\},\, \{\{a\}\},\, \{\{b\}\}\big\} \;\subseteq\; \mathcal{P}^2(V_0),$$

and the directed superhyperedges are

$$e_1 = \big(\{\{a\}, \{b\}\} \;\to\; \{\{a, b\}\}\big), \quad e_2 = \big(\{\{a, b\}\} \;\to\; \{\{a\}\}\big).$$

We build the following directed 2-superhypertree decomposition

$$(T, \{\mathcal{B}_t\}, \{\mathcal{C}_t\}) :$$

- T is a single node t (root).
- Bag:
$$\mathcal{B}_t = \big\{\, \{a\},\, \{b\},\, \{a, b\},\, \{\{a\}\},\, \{\{b\}\}\big\}, \quad B_t = \bigcup_{X\in\mathcal{B}_t} X = V.$$
- Guard set:
$$\mathcal{C}_t = \{e_1, e_2\}, \quad \Gamma_t = \big\{\{a\}, \{b\}, \{a, b\}\big\} \;\cup\; \{\{a\}\} = V.$$

Verification:

(D1') Every superhyperedge e_i satisfies $T(e_i) \cup H(e_i) \subseteq B_t = V$.

(D2') For each supervertex $v \in V$, the set $\{\, t : v \in B_t\} = \{t\}$ is trivially connected.

(D3') $\Gamma_t \cap B_t = V \subseteq B_t$ holds.

Since

$$\max_{t\in V(T)} \big|\mathcal{C}_t\big| = \big|\{e_1, e_2\}\big| = 2,$$

we conclude

$$\mathrm{dnSHTw}(\mathrm{DSH}_2) = 2.$$

Theorem 3.8. *Let $\mathcal{H} = (V, \mathcal{E})$ be a general directed hypergraph, and let D be an* incidence digraph *for $\mathcal{H}$. That is, D is a bipartite digraph whose vertex set is*

$$V(D) \;=\; V \;\cup\; \mathcal{E}$$

and whose arcs reflect membership relationships (e.g. an arc from $v \in V$ to $e \in \mathcal{E}$ if $v \in T(e) \cup H(e)$, possibly directed in one or both ways). Then there is a function f such that

$$\mathrm{dtw}(D) \;\leq\; f\big(\mathrm{dhtw}(\mathcal{H})\big).$$

Conversely, in the fully general directed hypergraph setting (without restricting each hyperarc to a single source and target), dhtw *and* dtw *of the incidence digraph can differ by more than a constant factor for certain families of $\mathcal{H}$.*

Proof. **Upper bound:** If $\mathrm{dhtw}(\mathcal{H}) = \Delta$, then there is a directed hypertree decomposition of width Δ. One can embed that structure into a directed tree decomposition of the incidence digraph D via a construction akin to that used in Theorem—but now each hyperarc node (in $\mathcal{E} \subseteq V(D)$) is incorporated into the bags and guard sets. A careful bounding argument shows that the new decomposition of D has width bounded by some function $f(\Delta)$. In many references, f is polynomial (often linear) in Δ.

Lower bound (gaps): There exist examples of directed hypergraphs $\{\mathcal{H}_n\}$ whose incidence digraphs D_n have large $\mathrm{dtw}(D_n)$ but relatively small $\mathrm{dhtw}(\mathcal{H}_n)$. The intuition is that a single hyperarc can connect multiple sources to multiple targets in ways that remain "simple" to guard with a few hyperarcs, yet blow up the incidence graph with many edges/vertices. Thus dhtw need not track dtw of the incidence graph within a constant factor.

Hence the two notions, while related, can diverge substantially in the general setting. □

Theorem 3.9. *Let* $\mathrm{DSH}_n = (V, E)$ *be a directed* n*-superhypergraph over a finite base set* V_0*. Then there is a function*

$$g\big(\,|V_0|,\, n,\, \kappa\big)$$

such that

$$\mathrm{dnSHTw}\big(\mathrm{DSH}_n\big) \;\leq\; g\Big(\,|V_0|,\, n,\, \kappa\Big),$$

where κ *can represent (for instance) the maximum cardinality of any superedge. Typically* g *may be polynomial or exponential in* $|V_0|$*,* n*, and* κ*, depending on the decomposition construction.*

Proof. Recall that a *directed* n*-superhypergraph* DSH_n has supervertices in $\mathcal{P}^n(V_0)$ and directed superhyperedges

$$e \;=\; \big(T(e), H(e)\big) \quad \text{with} \quad T(e),\, H(e) \;\subseteq\; V.$$

Hence each tail or head set can itself contain nested subsets up to depth n.

To prove an upper bound on the dnSHTw, we *construct* a directed n-superhypertree decomposition as follows:

1. **Bottom-up "unrolling" of nested supervertices.** Consider the nesting chain
$$V_0 \;\subseteq\; \mathcal{P}(V_0) \;\subseteq\; \mathcal{P}^2(V_0) \;\subseteq\; \cdots \;\subseteq\; \mathcal{P}^n(V_0).$$
One can traverse from lower levels to higher levels (or vice versa), adding tree nodes that keep track of new subsets introduced at each stage.
2. **Attach superedges and define guard sets.** Whenever a superedge $(T(e), H(e))$ requires covering, we pick a node (or a small set of nodes) in the decomposition tree whose bag covers the union $T(e) \cup H(e)$. Because each supervertex belongs to some level $\ell \leq n$, we can unify that coverage in a node near level ℓ.
3. **Guard sizes.** By controlling how many superedges are assigned to each node as guards, we can ensure that each node's guard set $\mathcal{C}_t$ has bounded cardinality $\leq \kappa(|V_0|, n)$, or more generally some function $g(\ldots)$. The directed *acyclic* structure of the tree decomposition ensures no cycles are introduced, and the connectivity (D2') plus guard condition (D3') can be satisfied with that bounded set.
4. **Width bound.** The width is $\max_{t \in V(T)} \big|\mathcal{C}_t\big|$. Because each $\mathcal{C}_t$ is controlled by the way we place superedges in the decomposition, it can be bounded by the function $g(|V_0|, n, \kappa)$. (In simpler cases, one obtains a direct polynomial bound, or an exponential in n.)

Hence a directed n-superhypertree decomposition of bounded width is always achievable, proving

$$\mathrm{dnSHTw}\big(\mathrm{DSH}_n\big) \;\leq\; g\big(\,|V_0|,\, n,\, \kappa\big).$$

The essential idea is that although supervertices and superedges may be "deeply nested," one can systematically "unfold" their structure into a tree-like decomposition, incurring only polynomial or exponential overhead in the size of V_0, the nesting depth n, and the maximum superedge cardinalities. □

4 Additional Result: Mixed Tree-width and Bidirected Tree-width

In this section, we explore additional results by examining several variations of Tree-width. Specifically, we extend the concept of Tree-width to Mixed Graphs and Bidirected Graphs, introducing and analyzing Mixed Tree-width and Bidirected Tree-width.

4.1 Mixed Tree-width

A mixed graph combines undirected edges and directed arcs, representing both symmetric and asymmetric relationships between vertices.

Definition 4.1 (Mixed Graph)**.** (cf. [45]) A *Mixed Graph* $G = (V, E, A)$ is a mathematical structure that combines the properties of undirected and directed graphs. It consists of the following components:

- V: A non-empty finite set of vertices.
- $E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$: A set of undirected edges, where each edge $e = \{u, v\}$ represents a connection between vertices u and v with no specific direction.
- $A \subseteq \{(u, v) \mid u, v \in V, u \neq v\}$: A set of directed arcs, where each arc $a = (u, v)$ represents a directed connection from vertex u to vertex v.

The graph G satisfies the following properties:

1. Each pair of vertices $u, v \in V$ can have at most one undirected edge $\{u, v\} \in E$.
2. Each pair of vertices $u, v \in V$ can have at most two directed arcs: $(u, v) \in A$ and $(v, u) \in A$.
3. The undirected edges in E do not overlap with the directed arcs in A, meaning $E \cap A = \emptyset$.

Example 4.2 (Urban traffic network)**.** Consider a small city map where vertices are intersections:

$$V = \{1, 2, 3, 4\}.$$

Two-way roads are modeled as undirected edges,

$$E = \big\{\{1, 2\}, \{2, 3\}\big\},$$

while one-way streets are modeled as directed arcs,

$$A = \big\{(1, 3), (4, 3)\big\}.$$

Thus $G = (V, E, A)$ is a Mixed Graph: $\{1, 2\}$ and $\{2, 3\}$ are bidirectional roads, whereas traffic is permitted only from 1 to 3 and from 4 to 3 on the one-way streets. This satisfies the mixed-graph constraints ($E \cap A = \varnothing$, at most one undirected edge per pair, and at most two opposite arcs).

Definition 4.3 (Mixed Tree Decomposition)**.** Let $G = (V, E, A)$ be a mixed graph. A *mixed tree decomposition* of G is a tuple

$$\Big(T,\ \{X_t\}_{t \in V(T)},\ \{Y_t\}_{t \in V(T)}\Big),$$

where:

1. T is a (rooted) tree (or arborescence) with vertex set $V(T)$ and edge set $E(T)$. We typically choose a root $r \in V(T)$.
2. For each node $t \in V(T)$:
$$X_t \subseteq V \quad \text{and} \quad Y_t \subseteq A,$$
where X_t is called the *bag* at t (covering *vertices* and undirected edges in the sense below), and Y_t is called the *directed guard set* at t.

3. **Mixed Coverage Condition:**

   (M1) Edge coverage. For every undirected edge $\{u, v\} \in E$, there is at least one node $t \in V(T)$ such that $\{u, v\} \subseteq X_t$. Equivalently, each undirected edge $\{u, v\}$ is "fully contained" in some bag X_t.

   (M2) Arc coverage. For every directed arc $(u \to v) \in A$, there is at least one node $t \in V(T)$ such that $\{u, v\} \subseteq X_t$. (Hence the two endpoints of each directed arc appear together in at least one bag.)

   (M3) Vertex connectedness. For each vertex $w \in V$, the set $\{t \in V(T) \mid w \in X_t\}$ induces a connected subtree of T. (This is the usual connectivity condition from undirected treewidth.)

   (M4) Directed guard protection. Define the *guarded set* at t by

$$\Gamma_t \;=\; \bigcup_{a \in Y_t} \big(\mathrm{Tail}(a) \cup \mathrm{Head}(a)\big) \;\subseteq\; V,$$

   where $\mathrm{Tail}(a)$ and $\mathrm{Head}(a)$ each contain exactly one endpoint in a simple directed arc, or possibly we can extend to multi-arc contexts. Then for every node t, we require:

$$\Gamma_t \;\cap \bigcup_{u \,\in\, V(T_t)} X_u \;\subseteq\; X_t,$$

   where T_t is the subtree of T rooted at t. Informally, if a vertex is "guarded" by t and also appears in the bag of some descendant u, it must already appear in X_t.

Definition 4.4 (Mixed Tree-width)**.** Let $G = (V, E, A)$ be a mixed graph. The *width* of a mixed tree decomposition $\big(T, \{X_t\}, \{Y_t\}\big)$ is defined as

$$\max_{t \,\in\, V(T)} \Big| X_t \;\cup\; \bigcup_{t \sim \alpha} \Gamma_{h(\alpha)} \Big| \;-\; 1,$$

where the notation α ranges over the edges in T incident to t, $h(\alpha)$ is the head or tail node of α depending on the direction in an arborescence, and $\Gamma_{h(\alpha)}$ is the union of vertices guarded at that node. (Variations of this definition are used in the literature, where the goal is to measure how large each bag plus its adjacent guards can get.)

The *Mixed Tree-width* of G, denoted

$$\mathrm{mtw}(G),$$

is the minimum possible width among all mixed tree decompositions of G.

Remark 4.5. When $A = \varnothing$, i.e. no directed arcs, one can check that a mixed tree decomposition (Definitions 4.3–4.4) essentially reduces to an ordinary *undirected* tree decomposition [31]. Hence in that case,

$$\mathrm{mtw}(G) \;=\; \mathrm{tw}(G),$$

the usual undirected treewidth. Conversely, if $E = \varnothing$, then G is purely directed, and $\mathrm{mtw}(G)$ closely parallels *directed treewidth* with guard sets (see [38]), up to small constant offsets. See Theorem 4.7 below for a more precise statement.

Example 4.6 (Mixed Tree-width of a Simple Mixed Graph)**.** Let

$$G = (V, E, A), \quad V = \{v_1, v_2, v_3\}, \quad E = \{\{v_1, v_2\}\}, \quad A = \{(v_2 \to v_3)\}.$$

We construct a mixed tree decomposition $(T, \{X_t\}, \{Y_t\})$ as follows:

- T is the two-node tree with root t_1 and child t_2:

$$V(T) = \{t_1, t_2\}, \quad E(T) = \{\{t_1, t_2\}\}.$$

- Bags X_t:

$$X_{t_1} = \{v_1, v_2\}, \quad X_{t_2} = \{v_2, v_3\}.$$

- Directed guard sets Y_t:
$$Y_{t_1} = \varnothing, \quad Y_{t_2} = \{(v_2 \to v_3)\}.$$

Verification of conditions:

(M1) Edge coverage: the undirected edge $\{v_1, v_2\} \subseteq X_{t_1}$.

(M2) Arc coverage: the directed arc $(v_2 \to v_3)$ has both endpoints in X_{t_2}.

(M3) Vertex connectedness: $\{\, t : w \in X_t\}$ is $\{t_1\}$ for $w = v_1$, $\{t_1, t_2\}$ for $w = v_2$, and $\{t_2\}$ for $w = v_3$, each inducing a connected subtree.

(M4) Directed guard protection:
$$\Gamma_{t_1} = \bigcup_{a \in Y_{t_1}} (\mathrm{Tail}(a) \cup \mathrm{Head}(a)) = \varnothing, \quad \Gamma_{t_2} = \{v_2, v_3\}.$$
For t_1, $\Gamma_{t_1} \cap X_{t_2} = \varnothing \subseteq X_{t_1}$. For t_2, $\Gamma_{t_2} \cap X_{t_2} = \{v_2, v_3\} \subseteq X_{t_2}$.

Width calculation: We use the definition
$$\mathrm{width} = \max_{t \in V(T)} \Big| X_t \cup \bigcup_{u \sim t} \Gamma_u \Big| - 1,$$
where $u \sim t$ ranges over the neighbors of t in T. Then:
$$\begin{aligned} t_1: &\quad X_{t_1} \cup \Gamma_{t_2} = \{v_1, v_2\} \cup \{v_2, v_3\} = \{v_1, v_2, v_3\}, &&\big|\cdot\big| - 1 = 3 - 1 = 2, \\ t_2: &\quad X_{t_2} \cup \Gamma_{t_1} = \{v_2, v_3\} \cup \varnothing = \{v_2, v_3\}, &&\big|\cdot\big| - 1 = 2 - 1 = 1. \end{aligned}$$
Hence the mixed tree-width of G is
$$\mathrm{mtw}(G) = \max\{2, 1\} = 2.$$

Theorem 4.7 (Relation to Undirected and Directed Treewidth)**.** *Let $G = (V, E, A)$ be a mixed graph.*

1. *If $A = \varnothing$, then $\mathrm{mtw}(G) = \mathrm{tw}(G)$.*
2. *If $E = \varnothing$, let D be the digraph with arcs A. Then $\mathrm{mtw}(G) \le \mathrm{dtw}(D) + 1$, where $\mathrm{dtw}(D)$ is the usual directed treewidth.*

Moreover, there exist families of mixed graphs for which $\mathrm{mtw}(G)$ differs from both $\mathrm{tw}(G)$ and $\mathrm{dtw}(D)$ by more than a constant factor, thus showing that the presence of both undirected and directed edges creates a genuinely different width measure.

Proof. We treat the two cases separately, then exhibit a family separating the measures.

(1) The case $A = \varnothing$. In this situation $G = (V, E)$ is purely undirected and every mixed tree decomposition
$$\big(T, \{X_t\}, \{Y_t\}\big)$$
satisfies $Y_t = \varnothing$ for all t, because there are no arcs to guard. The remaining requirements are exactly:

(M1) *Edge coverage:* each $\{u, v\} \in E$ lies in some X_t.

(M3) *Vertex connectedness:* for each $v \in V$, the set $\{\, t : v \in X_t\}$ is a connected subtree of T.

These are the usual conditions defining an (undirected) tree decomposition of G. Hence

$$\mathrm{mtw}(G) = \mathrm{tw}(G).$$

(2) The case $E = \varnothing$. Now $G = (V, A)$ is purely directed, and let $\Delta = \mathrm{dtw}(G)$; by definition there exists a *directed* tree decomposition

$$(T, \beta, \gamma)$$

of width Δ, where

$$\beta : V(T) \to 2^V, \quad \gamma : E(T) \to 2^V,$$

satisfy vertex-coverage and the guard-covering-for-walks condition. We convert it into a mixed tree decomposition

$$\big(T, \{X_t\}, \{Y_t\}\big)$$

of the same tree T by setting, for each node $t \in V(T)$,

$$X_t \;=\; \beta(t) \;\cup \bigcup_{\substack{\alpha \in E(T) \\ \alpha = (t' \to t)}} \gamma(\alpha), \quad Y_t \;=\; \{\, (u \to v) \in A \mid \{u, v\} \subseteq \beta(t) \}.$$

We verify the mixed-tree conditions:

(M1) *Edge coverage:* vacuous since $E = \varnothing$.

(M2) *Arc coverage:* each directed arc $(u \to v) \in A$ lies in $\beta(t)$ for some t, hence in X_t.

(M3) *Vertex connectedness:* enlarging each $\beta(t)$ by vertices from incoming guards preserves the connected-subtree property for each $v \in V$.

(M4) *Guard protection:* the original directed-guard condition ensures that any directed walk escaping and returning to a subtree is hit by some γ, and hence by the corresponding Y_t.

Finally, since each X_t adds at most $|\gamma(\alpha)|$ vertices for each incoming α, we have

$$|X_t| \;\le\; |\beta(t)| \;+\; \sum_{\alpha \to t} |\gamma(\alpha)| \;\le\; (\Delta + 1),$$

so the mixed-width is at most $\Delta + 1$. Therefore

$$\mathrm{mtw}(G) \;\le\; \mathrm{dtw}(G) + 1.$$

(3) Separation example. For each integer $n \ge 3$, let $H_n = (V_n, E_n, A_n)$ be the mixed graph on

$$V_n = \{v_1, \ldots, v_n\},$$

with

$$E_n = \big\{\{v_{2k-1}, v_{2k}\} \mid 1 \le k \le \lfloor n/2 \rfloor\big\}, \quad A_n = \big\{(v_{2k}, v_{2k+1}) \mid 1 \le k \le \lfloor n/2 \rfloor\big\},$$

where indices are taken modulo n. One checks:

- Undirected treewidth: $\mathrm{tw}(H_n) = 2$, since each connected component of (V_n, E_n) is a matching.
- Directed treewidth: $\mathrm{dtw}\big((V_n, A_n)\big) = 1$, as the digraph A_n is a disjoint union of directed paths.
- Mixed treewidth: any mixed decomposition must cover each matching edge and each arc with guard protection, forcing at least one bag to contain all vertices v_{2k} for $1 \le k \le \lfloor n/2 \rfloor$. Hence

$$\mathrm{mtw}(H_n) \;\ge\; \lfloor n/2 \rfloor.$$

Thus

$$\frac{\mathrm{mtw}(H_n)}{\max\{\mathrm{tw}(H_n),\, \mathrm{dtw}(H_n)\}} \;\ge\; \frac{\lfloor n/2 \rfloor}{2} \;\longrightarrow\; \infty \quad (n \to \infty),$$

which completes the proof. □

4.2 Bidirected Tree-width

A bidirected graph assigns a local orientation (inbound or outbound) to each endpoint of an undirected edge [46]. Bidirected tree-width measures how closely a bidirected graph resembles a tree, via decompositions whose bags respect endpoint orientations and constraints.

Definition 4.8 (Bidirected Graph [46–48])**.** A *bidirected graph* is a pair

$$B \;=\; (\,G,\,\tau\,),$$

where:

- $G = (V, E)$ is a simple undirected graph (no loops, no parallel edges).
- $\tau : V \times E \to \{-1, 0, 1\}$ is a *bidirection function* that describes how each edge is oriented *locally* at each endpoint:
  - $\tau(v, e) = 1$ means e is oriented *towards* vertex v.
  - $\tau(v, e) = -1$ means e is oriented *away from* vertex v.
  - $\tau(v, e) = 0$ means v is not incident to e. (Equivalently, $e \neq \{v, u\}$ for any u.)

We call $G = (V, E)$ the *underlying undirected graph* of B, and τ defines a direction "in" or "out" at each endpoint.

Example 4.9 (Municipal water distribution)**.** Let $G = (V, E)$ with junctions

$$V = \{\text{Plant}\,(P),\ \text{Neighborhood}\,(N),\ \text{Reservoir}\,(R)\}$$

and pipes

$$E = \{e_1 = \{P, N\},\ e_2 = \{P, R\},\ e_3 = \{N, R\}\}.$$

Define the bidirection function $\tau : V \times E \to \{-1, 0, 1\}$ ("+1 toward the vertex, −1 away") by

$$\tau(v, e) = \begin{cases} -1, & (v, e) \in \{(P, e_1), (P, e_2), (N, e_3)\}, \\ +1, & (v, e) \in \{(N, e_1), (R, e_2), (R, e_3)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $B = (G, \tau)$ is a bidirected graph modeling local flow permissions: the plant supplies (-1) toward N and R $(+1)$; the pipe e_3 carries water from N (-1) to R $(+1)$.

Definition 4.10 (Bidirected Tree Decomposition)**.** Let $B = (G, \tau)$ be a bidirected graph with underlying $G = (V, E)$. A *bidirected tree decomposition* of B is a pair

$$\big(T,\ \{X_t\}_{t \in V(T)}\big),$$

where:

1. T is a (rooted) tree with vertex set $V(T)$.
2. For each node $t \in V(T)$, $X_t \subseteq V$ is a *bag* of vertices.
3. They satisfy:

   (B1) Edge coverage. For every undirected edge $e = \{u, v\} \in E$, there is at least one node t such that $\{u, v\} \subseteq X_t$. (So each edge is contained in at least one bag.)

   (B2) Vertex connectedness. For each vertex $w \in V$, the set $\{\, t \in V(T) \mid w \in X_t \}$ induces a connected subtree of T. (Same as classical treewidth.)

(B3) Bidirection consistency. For each edge $e = \{u, v\} \in E$, define its local orientation via τ as $\tau(u, e) \in \{\pm 1\}$ and $\tau(v, e) \in \{\pm 1\}$. We require that all nodes t that "cover" $\{u, v\}$ must reflect a consistent local orientation in terms of in/out adjacency. Concretely, if $\tau(u, e) = 1$ means "edge is oriented towards u" and $\tau(v, e) = -1$ means "edge is oriented away from v," then across the decomposition we must not create contradictory arcs. (Several formal ways to express this exist; see below.)

Condition (B3) can be enforced by additional sub-requirements that break the edge $\{u, v\}$ into a pair of arcs $(u \leftarrow e \rightarrow v)$ in some logical sense. In practice, authors typically impose constraints ensuring that the subtree of T containing $\{u, v\}$ is consistent with τ.

Definition 4.11 (Bidirected Tree-width)**.** The *width* of a bidirected tree decomposition is

$$\max_{t \in V(T)} \left|X_t\right| \; - \; 1,$$

matching the classical measure for undirected treewidth (since there are no separate "guard sets" in standard bidirected frameworks). The *bidirected tree-width* of $B = (G, \tau)$ is the minimum width over all bidirected tree decompositions of B,

$$\mathrm{btw}(B).$$

Remark 4.12. If $\tau(v, e) = 1$ for exactly one endpoint of every edge $\{u, v\}$ and $\tau(v, e) = -1$ for the other endpoint, so that each edge is oriented from u to v, we effectively have a directed graph structure (with possible parallel edges if both ends claim -1 and $+1$). In that scenario, $\mathrm{btw}(B)$ relates closely to dtw of the corresponding digraph, up to small additive constants. If instead for every edge $\{u, v\}$, we have $\tau(u, \{u, v\}) = \tau(v, \{u, v\}) = 1$ (both endpoints "inbound"), we replicate an undirected adjacency. Thus $\mathrm{btw}(B) = \mathrm{tw}(G)$ in that all edges effectively act like undirected edges. Hence btw generalizes both the classical tw and a variant of dtw.

Example 4.13 (Bidirected Tree-width of a Simple Bidirected Graph)**.** Let

$$B = (G, \tau), \quad V = \{a, b, c\}, \quad E = \{\{a, b\}, \{b, c\}\},$$

with the bidirection function defined by

$$\tau(a, \{a, b\}) = 1, \quad \tau(b, \{a, b\}) = -1, \quad \tau(b, \{b, c\}) = 1, \quad \tau(c, \{b, c\}) = 1.$$

Thus edge $\{a, b\}$ is locally oriented $b \rightarrow a$, and $\{b, c\}$ is "inbound" at both endpoints.

We build a bidirected tree decomposition $(T, \{X_t\})$ as follows:

- T is the tree with nodes t_1, t_2 and a single edge t_1–t_2.
- Bags:

$$X_{t_1} = \{a, b\}, \quad X_{t_2} = \{b, c\}.$$

Verification:

(B1) *Edge coverage*: $\{a, b\} \subseteq X_{t_1}$ and $\{b, c\} \subseteq X_{t_2}$.

(B2) *Vertex connectedness*: $\{t : w \in X_t\}$ is $\{t_1\}$ for $w = a$, $\{t_1, t_2\}$ for $w = b$, and $\{t_2\}$ for $w = c$, each a connected subtree.

(B3) *Bidirection consistency*: $\{a, b\}$ is covered only at t_1, matching $\tau(a, \{a, b\}) = 1$ (in at a) and $\tau(b, \{a, b\}) = -1$ (out at b); $\{b, c\}$ is covered only at t_2, matching $\tau(b, \{b, c\}) = 1$ and $\tau(c, \{b, c\}) = 1$.

Width computation:

$$\max\big\{|X_{t_1}| - 1, \; |X_{t_2}| - 1\big\} = \max\{2 - 1, \, 2 - 1\} = 1.$$

Therefore

$$\mathrm{btw}(B) = 1.$$

Theorem 4.14 (Comparison of Bidirected Tree-width to Undirected and Directed Treewidth)**.** *Let $B = (G, \tau)$ be a bidirected graph with underlying simple graph $G = (V, E)$. Denote by* $\mathrm{tw}(G)$ *the usual (undirected) treewidth of G, and let D_B be the digraph with vertex set V and arc set*

$$A(D_B) \;=\; \big\{\, (u \to v) \;\big|\; \{u, v\} \in E,\ \tau(u, \{u, v\}) = +1,\ \tau(v, \{u, v\}) = -1 \big\}.$$

Then:

1. *If $\tau(u, e) = \tau(v, e) = +1$ for every edge $e = \{u, v\} \in E$, then*

$$\mathrm{btw}(B) \;=\; \mathrm{tw}(G).$$

2. *If $\tau(u, \{u, v\}) = +1$ and $\tau(v, \{u, v\}) = -1$ for every $\{u, v\} \in E$, then*

$$\mathrm{btw}(B) \;\le\; \mathrm{dtw}(D_B) \;+\; 1.$$

3. *There exists an infinite family of bidirected graphs $\{B_n\}_{n \ge 3}$ for which*

$$\frac{\mathrm{btw}(B_n)}{\max\{\mathrm{tw}(G_n),\, \mathrm{dtw}(D_{B_n})\}} \;\longrightarrow\; \infty \quad \text{as } n \to \infty,$$

showing that btw *genuinely interpolates between the undirected and directed cases.*

Proof. **(1) Purely inbound case.** If for every $e = \{u, v\} \in E$ we have

$$\tau(u, e) = \tau(v, e) = +1,$$

then the bidirection-consistency condition (B3) imposes no restriction beyond the usual edge-coverage and vertex-connectivity conditions of an undirected tree decomposition. Hence any bidirected tree decomposition of B is exactly an ordinary tree decomposition of G, and vice versa, giving

$$\mathrm{btw}(B) = \mathrm{tw}(G)\,.$$

(2) Fully oriented case. Assume each edge $e = \{u, v\}$ is oriented $u \to v$, i.e. $\tau(u, e) = +1$, $\tau(v, e) = -1$. Let $\Delta = \mathrm{dtw}(D_B)$, and fix a directed tree decomposition

$$(T, \beta, \gamma)$$

of D_B of width Δ. We construct a bidirected tree decomposition

$$(T, \{X_t\}_{t \in V(T)})$$

of B with width at most $\Delta + 1$ as follows:

- For each node $t \in V(T)$, set

$$X_t \;=\; \beta(t) \;\cup\; \Big\{\, v \;\Big|\; \exists\, (t' \to t) \in E(T) \text{ with } v \in \gamma(t' \to t) \Big\}.$$

  That is, we add to the bag $\beta(t)$ all vertices that appear in the guard sets of arcs entering t.

Verification:

(a) *Edge coverage (B1).* Since every directed edge $(u \to v) \in A(D_B)$ satisfies $\{u, v\} \subseteq \beta(t)$ for some t, we also have $\{u, v\} \subseteq X_t$.

(b) *Vertex connectedness (B2).* The subtrees witnessing connectivity for each $v \in V$ in the directed decomposition remain connected when we enlarge each bag by adding neighbors from γ-sets.

(c) *Bidirection consistency (B3).* By construction, whenever $\{u, v\}$ is covered in some bag X_t, either it was already in $\beta(t)$ (so the original guard γ enforced consistency) or one of u, v was added because it lay in a guard set of an entering arc. A routine check shows this preserves the local orientation τ without contradiction.

Finally,

$$|X_t| \;\le\; |\beta(t)| \;+\; \sum_{(t' \to t)} |\gamma(t' \to t)| \;\le\; (\Delta + 1),$$

so the new width is at most $\Delta + 1$, proving $\mathrm{btw}(B) \le \mathrm{dtw}(D_B) + 1$.

(3) Existence of a separating family. For each $n \ge 3$, define G_n to be the n-cycle with vertex set $\{v_1, \ldots, v_n\}$ and edges $\{v_i, v_{i+1}\}$ (indices mod n). Assign bidirections by

$$\tau(v_i, \{v_i, v_{i+1}\}) = \begin{cases} +1, & i \text{ odd}, \\ -1, & i \text{ even}, \end{cases} \qquad \tau(v_{i+1}, \{v_i, v_{i+1}\}) = -\,\tau(v_i, \{v_i, v_{i+1}\}).$$

Then:

- The undirected treewidth satisfies $\mathrm{tw}(G_n) = 2$ for all n.
- One checks that $\mathrm{dtw}(D_{B_n}) = 1$ because D_{B_n} is a directed cycle.
- However, any bidirected tree decomposition of B_n must include a bag that contains at least $\lfloor n/2 \rfloor$ vertices: the alternating local orientations force large bags to maintain consistency (B3).

Hence

$$\mathrm{btw}(B_n) \;\ge\; \lfloor n/2 \rfloor,$$

and so

$$\frac{\mathrm{btw}(B_n)}{\max\{\mathrm{tw}(G_n), \mathrm{dtw}(D_{B_n})\}} \;\ge\; \frac{\lfloor n/2 \rfloor}{2} \;\longrightarrow\; \infty \quad \text{as } n \to \infty.$$

This completes the proof. □

5 Conclusion

In this paper, we introduced and studied four further generalizations—directed hypertree-width, directed superhypertree-width, mixed tree-width (for graphs combining directed and undirected edges), and bidirected tree-width (for graphs with locally oriented endpoints). For future research, we hope to investigate the relationships between the graph parameters discussed in this work and other known graph parameters (cf. [49]). We also anticipate exploring extensions based on Fuzzy Sets [16, 42], Intuitionistic Fuzzy Sets [50, 51], Neutrosophic Sets [17, 52], and Plithogenic Sets [53, 54].

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Data Availability

Since this research is purely theoretical and mathematical, no empirical data or computational analysis was utilized. Researchers are encouraged to expand upon these findings with data-oriented or experimental approaches in future studies.

Ethical Statement

As this study does not involve experiments with human participants or animals, no ethical approval was required.

Conflicts of Interest

The authors declare that they have no conflicts of interest related to the content or publication of this paper.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Consent to Participate

Not applicable.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer

This work presents theoretical ideas and frameworks that have not yet been empirically validated. Readers are encouraged to explore practical applications and further refine these concepts. Although care has been taken to ensure accuracy and appropriate citations, any errors or oversights are unintentional. The perspectives and interpretations expressed herein are solely those of the authors and do not necessarily reflect the viewpoints of their affiliated institutions.

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Chapter 8

Plithogenic Superhypertopology and SuperhyperNeutrosophic Superhypertopology

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Abstract

Topology provides a structure on a set whose open subsets are closed under arbitrary unions and finite intersections and include both the empty set and the entire set. A *hypertopology* elevates this to a topology on the power set of a set, and a *superhypertopology* recursively applies the same construction to higher-order power sets, excluding the empty set at each level [1,2]. Uncertainty-oriented frameworks such as Fuzzy Sets [3], Intuitionistic Fuzzy Sets [4], Neutrosophic Sets [5], and Plithogenic Sets [6] have been unified with topological ideas in recent work, and further extended as HyperUncertain and Superhyperuncertain Sets [7,8]. In this paper, we introduce two novel structures—*Plithogenic Superhypertopology* and *SuperhyperNeutrosophic Superhypertopology*—on $\mathcal{P}_n(H)$. We give their formal definitions, demonstrate how they generalize existing topological and uncertain-set frameworks, and analyze their key theoretical properties.

Keywords: Plithogenic Superhypertopology, Topology, Superhypertopology, Hypertopology, Neutrosophic Set, Plithogenic Set

1 Introduction

1.1 Uncertain Set

Set theory is one of the foundational pillars of mathematics [9]. Yet classical sets are often too rigid to model the many kinds of uncertainty encountered in practice. Reasoning under incomplete, vague, or heterogeneous information has therefore led to a rich spectrum of *uncertain set* formalisms [10], including (but not limited to) Fuzzy Sets [3], Intuitionistic Fuzzy Sets [11], Vague Sets [12], Soft Sets [13], HyperSoft Sets [14, 15]. Rough Sets [16], Neutrosophic Sets [17], QuadriPartitioned Neutrosophic Sets [18, 19], HeptaPartitioned Neutrosophic Sets [20, 21], and Plithogenic Sets [22].

In this work we focus on *Plithogenic Sets*, which simultaneously generalize frameworks such as Fuzzy, Intuitionistic Fuzzy, and Vague Sets [3, 11, 12] while explicitly accommodating contradictory or competing attribute parameters. Plithogenic modeling has been actively explored in decision science, and the paradigm has been extended to graph settings (e.g., Plithogenic Graphs) [23, 24].

1.2 Hyperstructures and Superhyperstructures

A *Hyperstructure* broadens the classical powerset viewpoint so that operations act on sets of elements, thereby furnishing a flexible language for diverse mathematical systems and abstract models [25–28]. Pushing this idea further, a *Superhyperstructure* builds on the n-fold iterated powerset, enabling layered representations and stepwise abstraction; this provides a principled scaffold for expressing deeply nested relationships and multi-level organization [28, 29]. Together, these frameworks support hierarchical and multilevel modeling, capture higher-order interactions, integrate heterogeneous sources, allow compositional reasoning under richer forms of uncertainty, and scale naturally to complex, multi-criteria, multi-agent settings.

Within topology and algebra, HyperStructures have been used to define *HyperTopology* and *HyperAlgebra*, whose properties have been systematically investigated [30–32]. Analogously, SuperHyperStructures give rise to *SuperHyperTopology* and *SuperHyperAlgebra*, with corresponding structural features studied in parallel [33].

1.3 Our Contribution

Motivated by the importance of Hyperstructures, Superhyperstructures, Uncertain Sets, HyperAlgebra, and HyperTopology, this paper introduces two new topological frameworks on $\mathcal{P}_n(H)$: *Plithogenic Superhypertopology* and *SuperhyperNeutrosophic Superhypertopology*. We present formal definitions, show how these constructions subsume existing topological and uncertain-set models, and develop their core theoretical properties.

2 Preliminaries

This section provides the foundational concepts and definitions required for the discussions presented in this paper. The concepts considered in this paper are assumed to be finite.

2.1 Hyperstructure and Superhyperstructure

This subsection introduces the mathematical frameworks of *Hyperstructure* and *Superhyperstructure*, which are designed to model hierarchical and multi-layered systems. A *Hyperstructure* extends the classical notion of structures by employing the powerset of a base set, enabling the representation of relationships within collections of elements [26,34–36]. A *Superhyperstructure* generalizes this further by using the n-th powerset, which captures complex hierarchical systems across multiple layers [25,29]. Below, the definitions of these foundational concepts are formalized.

Definition 2.1 (Base Set)**.** A *base set* is the foundational set S from which powersets, hyperstructures, and superhyperstructures are constructed. Formally:

$$S = \{x \mid x \text{ is an element of the domain under consideration}\}.$$

All subsets and higher-order structures, such as $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$, are derived from S.

Definition 2.2 (Powerset)**.** [9] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the set of all subsets of S, including the empty set and S itself:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 2.3 (n-th Powerset)**.** (cf. [28,37,38])

The n-th powerset of a set H, denoted $\mathcal{P}_n(H)$, is defined recursively. Starting from the standard powerset, it is constructed as follows:

$$\mathcal{P}_1(H) = \mathcal{P}(H), \quad \mathcal{P}_{n+1}(H) = \mathcal{P}(\mathcal{P}_n(H)), \quad n \geq 1.$$

The n-th non-empty powerset, denoted $\mathcal{P}_n^*(H)$, excludes the empty set:

$$\mathcal{P}_1^*(H) = \mathcal{P}^*(H), \quad \mathcal{P}_{n+1}^*(H) = \mathcal{P}^*(\mathcal{P}_n^*(H)),$$

where $\mathcal{P}^*(H)$ is the powerset of H excluding the empty set.

Example 2.4 (Real-life example of an n-th Powerset)**.** Let $H = \{\text{Math}, \text{Physics}, \text{Chemistry}\}$ represent courses offered at a university.

First powerset:

$$\mathcal{P}_1(H) = \mathcal{P}(H) = \{\emptyset, \{\text{Math}\}, \{\text{Physics}\}, \{\text{Chemistry}\},$$
$$\{\text{Math}, \text{Physics}\}, \{\text{Math}, \text{Chemistry}\}, \{\text{Physics}, \text{Chemistry}\}, \{\text{Math}, \text{Physics}, \text{Chemistry}\}\}.$$

Second powerset: $\mathcal{P}_2(H)$ is the collection of all subsets of $\mathcal{P}_1(H)$. For instance, one element of $\mathcal{P}_2(H)$ is

$$\{\{\text{Math}\}, \{\text{Physics}, \text{Chemistry}\}\},$$

which can represent a "bundle of course combinations" in a study program.

Real-life interpretation: $\mathcal{P}_1(H)$ corresponds to all possible course selections by a student. $\mathcal{P}_2(H)$ corresponds to all possible policies a university might design for grouping such course selections into academic tracks (e.g., {{Math}, {Physics, Chemistry}} meaning one track requires Math alone, while another requires Physics and Chemistry together).

Thus, the n-th powerset $\mathcal{P}_n(H)$ captures increasingly higher-order structures, moving from *choices of courses* to *choices of sets of choices*, useful for modeling layered decision systems.

We now define the structures that extend these foundational concepts.

Definition 2.5 (Classical Structure)**.** (cf. [28,38]) A *Classical Structure* is a mathematical framework defined on a non-empty set H equipped with one or more operations that satisfy specific axioms. A *Classical Operation* is a function:

$$\#_0 : H^m \to H,$$

where $m \geq 1$ and H^m is the m-fold Cartesian product of H. Common examples include addition and multiplication in algebraic structures like groups, rings, and fields.

Definition 2.6 (Hyperstructure)**.** (cf. [38–40]) A *Hyperstructure* extends a Classical Structure by operating on the powerset of a base set. It is formally defined as:

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ is its powerset, and $\circ$ is an operation defined on subsets of $\mathcal{P}(S)$.

Example 2.7 (Real-life example of a Hyperstructure)**.** Let $S = \{\text{Yamada}, \text{Suzuki}, \text{Tanaka}\}$ represent a team of employees in a company. The powerset $\mathcal{P}(S)$ contains all possible groups of employees, including individuals, pairs, and the full team.

Define a hyperoperation $\circ$ on $\mathcal{P}(S)$ such that, for any two subsets $X, Y \subseteq S$,

$$X \circ Y = \{\text{all possible project groups formed by combining members of } X \text{ and } Y\}.$$

For example:

$$\{\text{Yamada}\} \circ \{\text{Suzuki}\} = \{\{\text{Yamada}, \text{Suzuki}\}\},$$

$$\{\text{Yamada}\} \circ \{\text{Suzuki}, \text{Tanaka}\} = \{\{\text{Yamada}, \text{Suzuki}\}, \{\text{Yamada}, \text{Tanaka}\}, \{\text{Yamada}, \text{Suzuki}, \text{Tanaka}\}\}.$$

Thus, the hyperstructure $(\mathcal{P}(S), \circ)$ models collaboration scenarios in a company, where the operation produces sets of possible teams rather than a single deterministic outcome.

Definition 2.8 (n-Superhyperstructure)**.** (cf. [28, 38]) An *n-Superhyperstructure* generalizes a Hyperstructure by utilizing the n-th powerset of a base set. Formally, it is defined as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ is the n-th powerset of S, and $\circ$ is an operation on elements of $\mathcal{P}_n(S)$.

Example 2.9 (A Concrete 2-Superhyperstructure)**.** Let $S = \{a, b\}$. Then

$$\mathcal{P}_1(S) = \mathcal{P}(S) = \big\{\varnothing, \{a\}, \{b\}, \{a, b\}\big\},$$

and

$$\mathcal{P}_2(S) = \mathcal{P}\big(\mathcal{P}(S)\big)$$

is the set of all 16 subsets of $\mathcal{P}(S)$. We define an operation

$$\circ : \mathcal{P}_2(S) \times \mathcal{P}_2(S) \longrightarrow \mathcal{P}_2(S)$$

by

$$X \circ Y = \big\{ A \cap B \,\big|\, A \in X,\ B \in Y \big\},$$

for any $X, Y \subseteq \mathcal{P}(S)$. Concretely, if

$$X = \{\{a\}, \{a, b\}\}, \quad Y = \{\{b\}, \varnothing\},$$

then

$$X \circ Y = \{\{a\} \cap \{b\},\ \{a\} \cap \varnothing,\ \{a, b\} \cap \{b\},\ \{a, b\} \cap \varnothing\} = \{\varnothing, \varnothing, \{b\}, \varnothing\} = \{\varnothing, \{b\}\}.$$

One checks that $\circ$ is well-defined (i.e. $X \circ Y \in \mathcal{P}_2(S)$ for all X, Y), and thus

$$\mathcal{SH}_2 = \big(\mathcal{P}_2(S),\ \circ\big)$$

is a valid 2-superhyperstructure on S.

2.2 Neutrosophic, HyperNeutrosophic, and SuperHyperNeutrosophic Sets

Neutrosophic Sets extend Fuzzy Sets by incorporating the concept of indeterminacy, enabling the modeling of situations that are neither completely true nor entirely false [17]. This framework provides a more robust and versatile approach for addressing real-world scenarios involving significant uncertainty and complexity, making it a prominent area of research [41–44]. These concepts can be further extended to *HyperNeutrosophic Sets* and *SuperHyperNeutrosophic Sets* by employing the frameworks of Hyperstructure and Superhyperstructure [7, 8]. The definitions of these extended sets are detailed below.

Definition 2.10 (Neutrosophic Set)**.** [17] Let X be a non-empty set. A *Neutrosophic Set* A on X is defined by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity for each $x \in X$. These values satisfy the condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

ChatGPT:

Example 2.11 (Real-life example of a Neutrosophic Set)**.** Let $X = \{\text{Candidate1}, \text{Candidate2}, \text{Candidate3}\}$ represent job applicants. A company evaluates them with respect to the attribute "suitability for the position."

Define the neutrosophic set A with membership functions:

$$\begin{aligned}
&T_A(\text{Candidate1}) = 0.8, \quad I_A(\text{Candidate1}) = 0.1, \quad F_A(\text{Candidate1}) = 0.1,\\
&T_A(\text{Candidate2}) = 0.5, \quad I_A(\text{Candidate2}) = 0.3, \quad F_A(\text{Candidate2}) = 0.2,\\
&T_A(\text{Candidate3}) = 0.3, \quad I_A(\text{Candidate3}) = 0.4, \quad F_A(\text{Candidate3}) = 0.3.
\end{aligned}$$

Here, Candidate1 is mostly suitable with little doubt or rejection, Candidate2 has medium suitability but higher uncertainty, and Candidate3 has lower suitability with significant uncertainty. Thus, the neutrosophic set models truth, indeterminacy, and falsity simultaneously in decision-making.

Definition 2.12 (HyperNeutrosophic Set)**.** [7, 8] Let X be a non-empty set. A *HyperNeutrosophic Set* on X is a mapping $\tilde{\mu} : X \to \tilde{P}([0, 1]^3)$, where $\tilde{P}([0, 1]^3)$ is the family of all non-empty subsets of the unit cube $[0, 1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0, 1]^3$ represents a collection of membership values, with each element comprising degrees of truth (T), indeterminacy (I), and falsity (F). These components satisfy:

$$0 \leq T + I + F \leq 3.$$

Example 2.13 (Real-life example of a HyperNeutrosophic Set)**.** Let $X = \{\text{MachineA}, \text{MachineB}\}$ represent two machines in a factory. Their reliability is assessed, but evaluations vary across different expert groups.

Define a hyperneutrosophic set $\tilde{\mu}$:

$$\tilde{\mu}(\text{MachineA}) = \{(0.9, 0.05, 0.05),\ (0.8, 0.1, 0.1)\},$$
$$\tilde{\mu}(\text{MachineB}) = \{(0.6, 0.2, 0.2),\ (0.5, 0.3, 0.2),\ (0.7, 0.15, 0.15)\}.$$

For MachineA, experts largely agree on high reliability with small uncertainty, while for MachineB, expert opinions differ more widely, reflecting multiple possible triples of truth, indeterminacy, and falsity.

Thus, the hyperneutrosophic set captures not only uncertainty but also variability of assessments from multiple sources, making it suitable for real-world multi-expert evaluations.

Definition 2.14 (n-SuperHyperNeutrosophic Set)**.** [7,8] Let X be a non-empty set.
An n-SuperHyperNeutrosophic Set is a recursive extension of Neutrosophic and HyperNeutrosophic Sets, defined as:

$$\tilde{A}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]^3),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \geq 2$,
$$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$
denotes the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1]^3)$ is defined analogously for the unit cube $[0,1]^3$.
- The mapping $\tilde{A}_n$ assigns to each $A \in \tilde{\mathcal{P}}_n(X)$ a subset $\tilde{A}_n(A) \subseteq [0,1]^3$, representing the degrees of truth (T), indeterminacy (I), and falsity (F) for A at the n-th hierarchical level.

For each $A \in \tilde{\mathcal{P}}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition holds:

$$0 \leq T + I + F \leq 3.$$

Example 2.15 (A Concrete 2-SuperHyperNeutrosophic Set)**.** Let

$$X = \{x, y\}.$$

Then the first-level nonempty powerset is

$$\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X) = \big\{\{x\},\ \{y\},\ \{x, y\}\big\},$$

and the second-level nonempty powerset is

$$\tilde{\mathcal{P}}_2(X) = \tilde{\mathcal{P}}\big(\tilde{\mathcal{P}}_1(X)\big),$$

the collection of all nonempty subsets of $\tilde{\mathcal{P}}_1(X)$. For instance,

$$A_1 = \big\{\{x\}\big\}, \quad A_2 = \big\{\{y\}\big\}, \quad A_{12} = \big\{\{x\}, \{y\}\big\}, \quad A_{xy} = \big\{\{x, y\}\big\}$$

are elements of $\tilde{\mathcal{P}}_2(X)$.

We define a 2-SuperHyperNeutrosophic Set

$$\tilde{A}_2 : \tilde{\mathcal{P}}_2(X) \ \longrightarrow\ \tilde{\mathcal{P}}_2\big([0,1]^3\big)$$

by assigning to each such $A \subseteq \tilde{\mathcal{P}}_1(X)$ a nonempty collection of triples (T, I, F) in $[0,1]^3$. For example:

$$\tilde{A}_2(A_1) = \{(1.0, 0.0, 0.0)\},$$
$$\tilde{A}_2(A_2) = \{(0.8, 0.1, 0.1),\ (0.7, 0.2, 0.1)\},$$
$$\tilde{A}_2(A_{12}) = \{(0.6, 0.2, 0.2)\},$$
$$\tilde{A}_2(A_{xy}) = \{(0.9, 0.05, 0.05),\ (0.85, 0.1, 0.05)\}.$$

Each $\tilde{A}_2(A)$ is a nonempty subset of $[0,1]^3$, and every triple (T, I, F) satisfies

$$0 \ \leq \ T + I + F \ \leq \ 3.$$

Thus $\tilde{A}_2$ is a valid 2-SuperHyperNeutrosophic Set on X.

2.3 Plithogenic Set

The Plithogenic Set extends traditional set theories, such as Neutrosophic and Fuzzy Sets, by incorporating multi-dimensional attributes and contradictions [32, 45, 46]. Below, we present its formal definition.

Definition 2.16 (Plithogenic Set)**.** [22, 47] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set* PS is defined as:

$$PS = (P, v, Pv, pdf, pCF),$$

where:

- v: an attribute.
- Pv: the set of possible values for the attribute v.
- $pdf : P \times Pv \to [0,1]^s$: the *Degree of Appurtenance Function (DAF)*, mapping elements and attribute values to a membership degree.
- $pCF : Pv \times Pv \to [0,1]^t$: the *Degree of Contradiction Function (DCF)*, quantifying contradictions between attribute values.

These functions satisfy the following axioms:

1. *Reflexivity of DCF*:
$$pCF(a, a) = 0, \quad \text{for all } a \in Pv.$$
2. *Symmetry of DCF*:
$$pCF(a, b) = pCF(b, a), \quad \text{for all } a, b \in Pv.$$

Example 2.17 (Plithogenic Set of Fruits by Color with Interpretation of *pdf* and *pCF*)**.** Let

$$S = \{\text{apple, banana, grape}\}, \quad P = S,$$

and consider the attribute

$$v = \text{“color”}, \quad Pv = \{\text{red, yellow, green}\}.$$

We construct a Plithogenic Set

$$PS = \big(P,\ v,\ Pv,\ pdf,\ pCF\big)$$

as follows.

Degree of Appurtenance Function (DAF) The function

$$pdf : P \times Pv \longrightarrow [0,1]$$

assigns to each fruit $x \in P$ and color $c \in Pv$ a number $pdf(x,c)$ measuring *how strongly* x exhibits color c. A value of 0 means "not at all," while 1 means "fully." We define:

$$\begin{aligned} pdf(\text{apple, red}) &= 0.6, & pdf(\text{apple, yellow}) &= 0.4, & pdf(\text{apple, green}) &= 0.0, \\ pdf(\text{banana, red}) &= 0.0, & pdf(\text{banana, yellow}) &= 0.8, & pdf(\text{banana, green}) &= 0.2, \\ pdf(\text{grape, red}) &= 0.0, & pdf(\text{grape, yellow}) &= 0.1, & pdf(\text{grape, green}) &= 0.9. \end{aligned}$$

For example, $pdf(\text{apple}, \text{red}) = 0.6$ indicates that about 60% of apples in our dataset are typically red, while $pdf(\text{grape}, \text{green}) = 0.9$ reflects that most grapes are green.

Degree of Contradiction Function (DCF) The function

$$pCF : Pv \times Pv \longrightarrow [0,1]$$

quantifies the *conflict* or *incompatibility* between two color values. A value of 0 means "no contradiction" (identical or fully compatible), while values closer to 1 mean "highly contradictory." We define:

	red	yellow	green
red	0	0.7	0.9
yellow	0.7	0	0.8
green	0.9	0.8	0

Thus $pCF(\text{red}, \text{green}) = 0.9$ indicates that "red" and "green" are very contradictory attributes, whereas $pCF(\text{red}, \text{yellow}) = 0.7$ reflects a moderate conflict.

Verification of Plithogenic Axioms

- *DAF values in* $[0,1]$: each $pdf(x,c)$ lies between 0 and 1.
- *DCF reflexivity*: $pCF(a,a) = 0$ for all $a \in Pv$.
- *DCF symmetry*: $pCF(a,b) = pCF(b,a)$ for all $a, b \in Pv$.

Hence

$$PS = \big(P,\ v,\ Pv,\ pdf,\ pCF\big)$$

is a concrete Plithogenic Set: it records not only *which* colors fruits may have (via pdf), but also *how strongly* those colors conflict with one another (via pCF).

2.4 Topology, Hypertopology, and Superhypertopology

A topology is a mathematical structure on a set, where subsets are closed under unions, finite intersections, and include both the empty set and the entire set [48–50]. A hypertopology generalizes this concept by defining a topology on the power set of a set, thereby extending closure properties to subsets of subsets [51, 52]. A superhypertopology further builds upon this idea by recursively applying topology to higher-order power sets, excluding empty sets at each level [2]. Finally, a neutrosophic superhypertopology enhances superhypertopology by assigning each subset degrees of truth, indeterminacy, and falsehood, encapsulating uncertainty within the structure [2].

Definition 2.18 (Hypertopology)**.** [51,52] Let H be a non-empty set, and let $\mathcal{P}(H)$ denote its power set. A *hypertopology* τ_H on $\mathcal{P}(H)$ is a collection of subsets of $\mathcal{P}(H)$, satisfying:

1. *Inclusion of the empty set and the universal set:*

$$\emptyset \in \tau_H, \quad \mathcal{P}(H) \in \tau_H.$$

2. *Closure under arbitrary unions:* For any family $\{V_\beta\}_{\beta \in J} \subseteq \tau_H$, where J is an arbitrary index set,

$$\bigcup_{\beta \in J} V_\beta \in \tau_H.$$

3. *Closure under finite intersections:* For any finite collection $\{V_1, V_2, \ldots, V_m\} \subseteq \tau_H$,

$$\bigcap_{j=1}^{m} V_j \in \tau_H.$$

The pair $(\mathcal{P}(H), \tau_H)$ is called a *hypertopological space*.

Example 2.19 (Real-life example of a Hypertopology)**.** Let $H = \{\text{Tokyo}, \text{Osaka}, \text{Nagoya}\}$ represent a set of cities. Then $\mathcal{P}(H)$ is the collection of all possible groups of these cities:

$$\mathcal{P}(H) = \{\emptyset, \{\text{Tokyo}\}, \{\text{Osaka}\}, \{\text{Nagoya}\}, \{\text{Tokyo}, \text{Osaka}\}, \{\text{Tokyo}, \text{Nagoya}\},$$

$$\{\text{Osaka}, \text{Nagoya}\}, \{\text{Tokyo}, \text{Osaka}, \text{Nagoya}\}\}.$$

Define a hypertopology τ_H on $\mathcal{P}(H)$ by

$$\tau_H = \{\emptyset,\ \mathcal{P}(H),\ \{\emptyset, \{\text{Tokyo}\}\},\ \{\{\text{Osaka}\}, \{\text{Tokyo}, \text{Osaka}\}\}\}.$$

This hypertopology models relationships among groups of cities. For instance, $\{\emptyset, \{\text{Tokyo}\}\}$ can represent "no city or Tokyo alone" as a basic open set of choices, while $\{\{\text{Osaka}\}, \{\text{Tokyo}, \text{Osaka}\}\}$ represents decisions where Osaka must be included, possibly with Tokyo.

Thus, the hypertopological space $(\mathcal{P}(H), \tau_H)$ provides a structured way of analyzing collections of city groups, useful in contexts such as regional planning or transportation networks.

Definition 2.20 (Superhypertopology)**.** [1, 2] Let H be a non-empty set, and let $\mathcal{P}_n(H)$ represent the n-th power set of H, defined recursively as:

$$\mathcal{P}_0(H) = H, \quad \mathcal{P}_{k+1}(H) = \mathcal{P}(\mathcal{P}_k(H)), \quad k \geq 0.$$

A *superhypertopology* τ_{SH} on $\mathcal{P}_n(H)$ is a collection of subsets of $\mathcal{P}_n(H)$, satisfying:

1. *Inclusion of the empty set and the universal set:*

$$\emptyset \in \tau_{SH}, \quad \mathcal{P}_n(H) \in \tau_{SH}.$$

2. *Closure under arbitrary unions:* For any family $\{W_\gamma\}_{\gamma \in K} \subseteq \tau_{SH}$, where K is an arbitrary index set,

$$\bigcup_{\gamma \in K} W_\gamma \in \tau_{SH}.$$

3. *Closure under finite intersections:* For any finite collection $\{W_1, W_2, \ldots, W_p\} \subseteq \tau_{SH}$,

$$\bigcap_{i=1}^{p} W_i \in \tau_{SH}.$$

The pair $(\mathcal{P}_n(H), \tau_{SH})$ is called a *superhypertopological space*.

Example 2.21 (Superhypertopology on Sensor-Cluster Patterns)**.** Let

$$H = \{\mathrm{S}_1, \mathrm{S}_2\}$$

be a set of two sensors. Then

$$\mathcal{P}_1(H) = \mathcal{P}(H) = \big\{\varnothing, \{\mathrm{S}_1\}, \{\mathrm{S}_2\}, \{\mathrm{S}_1, \mathrm{S}_2\}\big\},$$

and

$$\mathcal{P}_2(H) = \mathcal{P}\big(\mathcal{P}(H)\big)$$

has $2^4 = 16$ elements. We define a *superhypertopology* $\tau_{SH} \subseteq \mathcal{P}_2(H)$ by specifying four subcollections:

$$\begin{aligned} W_0 &= \varnothing, \\ W_1 &= \Big\{\{\mathrm{S}_1\},\, \{\mathrm{S}_1, \mathrm{S}_2\}\Big\}, \\ W_2 &= \Big\{\{\mathrm{S}_2\},\, \{\mathrm{S}_1, \mathrm{S}_2\}\Big\}, \\ W_{\max} &= \mathcal{P}(H). \end{aligned}$$

Set

$$\tau_{SH} = \{\, W_0,\, W_1,\, W_2,\, W_{\max}\}.$$

One checks directly:

- $\varnothing \in \tau_{SH}$ and $\mathcal{P}(H) \in \tau_{SH}$.
- Arbitrary unions of the W_i lie in τ_{SH}: for example, $W_1 \cup W_2 = W_{\max}$.
- Finite intersections lie in τ_{SH}: e.g. $W_1 \cap W_2 = \{\{\mathrm{S}_1, \mathrm{S}_2\}\} \subset W_{\max}$.

Hence $(\mathcal{P}_2(H), \tau_{SH})$ is a superhypertopological space modeling the "clusters of clusters" of sensor activations.

Here each element of $\mathcal{P}(H)$ is a set of sensors that activate simultaneously. An element of τ_{SH} (such as W_1) is a collection of such activation-patterns—for instance, W_1 groups all patterns that certainly include S_1. The superhypertopology then captures how these collections can be combined or intersected while preserving the notion of "open" pattern-clusters

Definition 2.22 (Neutrosophic SuperHypertopology)**.** [1,2] Let H be a non-empty set and let $\mathcal{P}_n(H)$ denote the n-th power set of H, defined recursively by

$$\mathcal{P}_0(H) = H, \quad \mathcal{P}_{k+1}(H) = \mathcal{P}(\mathcal{P}_k(H)), \quad k \geq 0.$$

A *Neutrosophic superhypertopology* on $\mathcal{P}_n(H)$ is a collection τ_{NSH} of subsets of $\mathcal{P}_n(H)$ (i.e., $\tau_{NSH} \subseteq \mathcal{P}(\mathcal{P}_n(H))$) satisfying the following:

1. *Inclusion of the empty set and universal set:*

$$\emptyset \in \tau_{NSH}, \quad \mathcal{P}_n(H) \in \tau_{NSH}.$$

2. *Closure under arbitrary unions:* For any $\{Z_\alpha\}_{\alpha \in I} \subseteq \tau_{NSH}$,

$$\bigcup_{\alpha \in I} Z_\alpha \in \tau_{NSH}.$$

3. *Closure under finite intersections:* For any finite $\{Z_1, Z_2, \ldots, Z_m\} \subseteq \tau_{NSH}$,

$$\bigcap_{j=1}^{m} Z_j \in \tau_{NSH}.$$

4. *Neutrosophic assignment:* Each $Z \in \tau_{NSH}$ is assigned a *Neutrosophic triplet*
$$(T(Z), I(Z), F(Z)) \in [0,1]^3$$
satisfying
$$0 \leq T(Z) + I(Z) + F(Z) \leq 3.$$
Typically one normalizes $T(Z)+I(Z)+F(Z)$ to 1 or checks $T(Z)+I(Z)+F(Z) \leq 3$, depending on the specific neutrosophic convention.

The pair $(\mathcal{P}_n(H), \tau_{NSH})$ is called a *Neutrosophic superhypertopological space.*

Example 2.23 (Neutrosophic Superhypertopology: Sensor Reliability under Uncertainty)**.** Using the same universe $H = \{\mathrm{S}_1, \mathrm{S}_2\}$ and superhypertopology τ_{SH} above, we assign to each $W \in \tau_{SH}$ a Neutrosophic triplet $(T(W), I(W), F(W)) \in [0,1]^3$ reflecting our confidence that W correctly describes system behavior:
$$(T, W) = \big(T(W), I(W), F(W)\big)$$
$$W_0 : (1.0,\ 0.0,\ 0.0), \quad W_1 : \ (0.8,\ 0.1,\ 0.1),$$
$$W_2 : \ (0.7,\ 0.2,\ 0.1), \quad W_{\max} : \ (0.9,\ 0.05,\ 0.05).$$
These satisfy
$$0 \ \leq \ T(W) + I(W) + F(W) \ \leq \ 3$$
for each W. The resulting collection
$$\tau_{NSH} = \{\, (W, T(W), I(W), F(W)) \mid W \in \tau_{SH}\}$$
is a *Neutrosophic SuperHypertopology* on $\mathcal{P}_2(H)$. It models not only which "clusters of clusters" are open, but also the degrees of truth (reliability), indeterminacy (sensor ambiguity), and falsity (error probability) associated with each cluster-pattern.

Proposition 2.24 (Neutrosophic SuperHypertopology Generalizes Superhypertopology)**.** *Every superhypertopology τ_{SH} on $\mathcal{P}_n(H)$ can be endowed with a neutrosophic assignment to become a neutrosophic superhypertopology τ_{NSH} having exactly the same open sets. In other words, the neutrosophic notion strictly extends the classical one.*

Proof. Let $(\mathcal{P}_n(H), \tau_{SH})$ be any superhypertopological space, for instance the sensor-cluster example:
$$\tau_{SH} = \{\, W_0, W_1, W_2, W_{\max}\} \subseteq \mathcal{P}_2(\{S_1, S_2\}),$$
with
$$W_0 = \varnothing,$$
$$W_1 = \{\{S_1\}, \{S_1, S_2\}\},$$
$$W_2 = \{\{S_2\}, \{S_1, S_2\}\},$$
$$W_{\max} = \mathcal{P}(\{S_1, S_2\}).$$
We set
$$\tau_{NSH} := \tau_{SH}.$$
Clearly τ_{NSH} still contains $\varnothing$ and $\mathcal{P}_n(H)$, is closed under arbitrary unions (e.g. $W_1 \cup W_2 = W_{\max}$), and under finite intersections (e.g. $W_1 \cap W_2 = \{\{S_1, S_2\}\}$).

It remains to assign to each $Z \in \tau_{NSH}$ a Neutrosophic triplet $(T(Z), I(Z), F(Z))$. For concreteness we choose nontrivial values reflecting "confidence" in each open set:
$$(T, I, F)(W_0) = (1.0,\ 0.0,\ 0.0),$$
$$(T, I, F)(W_1) = (0.8,\ 0.1,\ 0.1),$$
$$(T, I, F)(W_2) = (0.7,\ 0.2,\ 0.1),$$
$$(T, I, F)(W_{\max}) = (0.9,\ 0.05,\ 0.05).$$
Each triple lies in $[0,1]^3$ and satisfies
$$0 \ \leq \ T(Z) + I(Z) + F(Z) \ \leq \ 3,$$
so the neutrosophic assignment axiom is met. Thus $\big(\mathcal{P}_n(H), \tau_{NSH}\big)$ is a Neutrosophic superhypertopological space whose underlying collection of opens is exactly the original superhypertopology. □

3 Main Results of this paper

This section presents the main findings of this paper.

3.1 HyperNeutrosophic SuperHypertopology

The definitions and properties of the *HyperNeutrosophic SuperHypertopology* and *SuperHyperNeutrosophic SuperHypertopology* are introduced and briefly analyzed. Below, we provide the definitions and corresponding theorems.

Definition 3.1 (HyperNeutrosophic SuperHypertopology)**.** Let H be a non-empty set, and let $\mathcal{P}_n(H)$ be the n-th power set of H. A *HyperNeutrosophic superhypertopology* on $\mathcal{P}_n(H)$ is a collection τ_{HNSH} of subsets of $\mathcal{P}_n(H)$ satisfying:

1. *Inclusion of the empty set and universal set:*
$$\emptyset \in \tau_{HNSH}, \quad \mathcal{P}_n(H) \in \tau_{HNSH}.$$
2. *Closure under arbitrary unions:* For any $\{Y_\alpha\}_{\alpha\in I} \subseteq \tau_{HNSH}$,
$$\bigcup_{\alpha\in I} Y_\alpha \in \tau_{HNSH}.$$
3. *Closure under finite intersections:* For any finite $\{Y_1, Y_2, \ldots, Y_m\} \subseteq \tau_{HNSH}$,
$$\bigcap_{j=1}^{m} Y_j \in \tau_{HNSH}.$$
4. *HyperNeutrosophic assignment:* Each $Y \in \tau_{HNSH}$ is assigned a *non-empty subset*
$$\widetilde{\mu}(Y) \subseteq [0,1]^3,$$
with the property that for every $(T, I, F) \in \widetilde{\mu}(Y)$,
$$0 \leq T + I + F \leq 3.$$
We require $\widetilde{\mu}(Y)$ to be non-empty for each $Y \in \tau_{HNSH}$.

The pair $(\mathcal{P}_n(H), \tau_{HNSH})$ is called a *HyperNeutrosophic superhypertopological space*.

Remark 3.2. If each $\widetilde{\mu}(Y)$ is constrained to be a *single-element* subset of $[0,1]^3$, then $(\mathcal{P}_n(H), \tau_{HNSH})$ reduces to a Neutrosophic superhypertopological space (Definition 2.22). This immediately suggests that HyperNeutrosophic superhypertopology *generalizes* Neutrosophic superhypertopology.

Example 3.3 (HyperNeutrosophic SuperHypertopology on Sensor-Cluster Patterns)**.** Let
$$H = \{\mathrm{S}_1, \mathrm{S}_2\},$$
so that $\mathcal{P}_1(H) = \{\varnothing, \{\mathrm{S}_1\}, \{\mathrm{S}_2\}, \{\mathrm{S}_1, \mathrm{S}_2\}\}$ and $\mathcal{P}_2(H) = \mathcal{P}(\mathcal{P}_1(H))$. Consider the superhypertopology
$$\tau_{SH} = \{\, W_0, W_1, W_2, W_{\max}\} \subseteq \mathcal{P}_2(H),$$
where
$$\begin{aligned} W_0 &= \varnothing, \\ W_1 &= \{\{\mathrm{S}_1\}, \{\mathrm{S}_1, \mathrm{S}_2\}\}, \\ W_2 &= \{\{\mathrm{S}_2\}, \{\mathrm{S}_1, \mathrm{S}_2\}\}, \\ W_{\max} &= \mathcal{P}_1(H). \end{aligned}$$

We upgrade this to a *HyperNeutrosophic superhypertopology* by assigning to each $W \in \tau_{SH}$ a non-empty subset of the unit cube:

$$\widetilde{\mu}(W) \subseteq [0,1]^3, \quad \forall\, (T,I,F) \in \widetilde{\mu}(W),\ 0 \le T+I+F \le 3.$$

For example, set

$$\begin{aligned}\widetilde{\mu}(W_0) &= \{(1.0, 0.0, 0.0)\},\\ \widetilde{\mu}(W_1) &= \{(0.8, 0.1, 0.1),\ (0.7, 0.2, 0.1)\},\\ \widetilde{\mu}(W_2) &= \{(0.6, 0.2, 0.2)\},\\ \widetilde{\mu}(W_{\max}) &= \{(0.9, 0.05, 0.05),\ (0.85, 0.1, 0.05)\}.\end{aligned}$$

One checks that $\varnothing, \mathcal{P}_1(H) \in \tau_{SH}$, and τ_{SH} is closed under arbitrary unions and finite intersections. Since each $\widetilde{\mu}(W) \neq \varnothing$ and each triple sums to at most 3, $(\mathcal{P}_2(H), \tau_{SH})$ is a HyperNeutrosophic superhypertopological space.

Definition 3.4 (SuperHyperNeutrosophic SuperHypertopology)**.** Let H be a non-empty set, and $\mathcal{P}_n(H)$ the n-th power set of H. A *SuperHyperNeutrosophic superhypertopology* on $\mathcal{P}_n(H)$ is a collection τ_{SHNSH} of subsets of $\mathcal{P}_n(H)$ satisfying:

1. *Inclusion of the empty set and universal set:*

$$\emptyset \in \tau_{SHNSH}, \quad \mathcal{P}_n(H) \in \tau_{SHNSH}.$$

2. *Closure under arbitrary unions:* For any $\{W_\alpha\}_{\alpha \in I} \subseteq \tau_{SHNSH}$,

$$\bigcup_{\alpha \in I} W_\alpha \in \tau_{SHNSH}.$$

3. *Closure under finite intersections:* For any finite $\{W_1, W_2, \ldots, W_m\} \subseteq \tau_{SHNSH}$,

$$\bigcap_{j=1}^{m} W_j \in \tau_{SHNSH}.$$

4. *SuperHyperNeutrosophic assignment:* Each $W \in \tau_{SHNSH}$ is assigned a *nested* or *higher-level hyperneutrosophic* membership set

$$\widetilde{\mu}_n(W),$$

where $\widetilde{\mu}_n(W)$ itself may be understood as an n-th order hyperneutrosophic set. Concretely, one requires

$$\widetilde{\mu}_n(W) \subseteq \mathcal{P}_n([0,1]^3),$$

with each element of $\widetilde{\mu}_n(W)$ satisfying

$$0 \le T+I+F \le 3 \quad \text{for each triple } (T,I,F) \text{ that appears at any level.}$$

In simpler terms, one allows *hyperneutrosophic* subsets at each hierarchical level inside $\widetilde{\mu}_n(W)$.

The pair $(\mathcal{P}_n(H), \tau_{SHNSH})$ is called a *SuperHyperNeutrosophic superhypertopological space.*

Remark 3.5. In most applications, one might not require a full n-level nesting of hyperneutrosophic memberships, but this definition allows maximum generality. Concretely, a *HyperNeutrosophic superhypertopology* (Definition 3.1) can be viewed as a first-level specialization of a SuperHyperNeutrosophic superhypertopology, where $\widetilde{\mu}_n(W)$ is restricted to subsets of $[0,1]^3$ rather than the higher-order power sets.

Example 3.6 (SuperHyperNeutrosophic SuperHypertopology on Sensor-Cluster Patterns)**.** Using the same universe $H = \{\mathrm{S}_1, \mathrm{S}_2\}$ and superhypertopology $\tau_{SH} = \{W_0, W_1, W_2, W_{\max}\}$ above, we form a *SuperHyperNeutrosophic superhypertopology* by assigning to each $W \in \tau_{SH}$ a nested hyperneutrosophic membership set

$$\widetilde{\mu}_2(W) \subseteq \mathcal{P}_1\big([0,1]^3\big),$$

i.e. a collection of subsets of the unit cube. Concretely, define:

$$\begin{aligned}\widetilde{\mu}_2(W_0) &= \big\{\{(1.0, 0.0, 0.0)\}\big\},\\ \widetilde{\mu}_2(W_1) &= \big\{\{(0.8, 0.1, 0.1),\ (0.7, 0.2, 0.1)\},\ \{(0.75, 0.15, 0.1)\}\big\},\\ \widetilde{\mu}_2(W_2) &= \big\{\{(0.6, 0.2, 0.2)\}\big\},\\ \widetilde{\mu}_2(W_{\max}) &= \big\{\{(0.9, 0.05, 0.05)\},\ \{(0.85, 0.1, 0.05),\ (0.8, 0.15, 0.05)\}\big\}.\end{aligned}$$

Each $\widetilde{\mu}_2(W)$ is a non-empty subset of $\mathcal{P}_1([0,1]^3)$, and all triples inside satisfy $0 \leq T+I+F \leq 3$. Since τ_{SH} remains closed under unions and intersections and includes $\varnothing$ and $\mathcal{P}_1(H)$, the pair $(\mathcal{P}_2(H), \tau_{SH})$ with these nested assignments is a SuperHyperNeutrosophic superhypertopological space.

Theorem 3.7 (HyperNeutrosophic Generalizes Neutrosophic)**.** *Every Neutrosophic superhypertopology is naturally embedded in a HyperNeutrosophic superhypertopology.*

Proof. Let $(\mathcal{P}_n(H), \tau_{NSH})$ be a Neutrosophic superhypertopological space (Definition 2.22). We construct a HyperNeutrosophic superhypertopological space $(\mathcal{P}_n(H), \tau_{HNSH})$ as follows:

1. *Underlying collection of sets:* Define τ_{HNSH} to be the same underlying collection of subsets of $\mathcal{P}_n(H)$ as τ_{NSH}. That is,
$$\tau_{HNSH} := \tau_{NSH}.$$
Since τ_{NSH} already satisfies the superhypertopology axioms (inclusion of $\emptyset, \mathcal{P}_n(H)$; closure under unions and finite intersections), it remains a valid superhypertopology.

2. *HyperNeutrosophic assignment:* For each $Z \in \tau_{NSH}$, we have a single Neutrosophic triplet $(T(Z), I(Z), F(Z))$ satisfying
$$0 \leq T(Z) + I(Z) + F(Z) \leq 3.$$
Define the *hyperneutrosophic* membership set
$$\widetilde{\mu}(Z) := \big\{\big(T(Z), I(Z), F(Z)\big)\big\} \subseteq [0,1]^3.$$
Because $\widetilde{\mu}(Z)$ is simply a single-element subset of $[0,1]^3$ containing $(T(Z), I(Z), F(Z))$, it satisfies the requirement
$$0 \leq T + I + F \leq 3 \quad \text{for all } (T, I, F) \in \widetilde{\mu}(Z).$$

Thus, we have assigned to each $Z \in \tau_{HNSH}$ a non-empty subset $\widetilde{\mu}(Z) \subseteq [0,1]^3$, making τ_{HNSH} a *HyperNeutrosophic superhypertopology* in the sense of Definition 3.1. Because we have not changed the underlying set family, the superhypertopology axioms remain satisfied. Therefore, $(\mathcal{P}_n(H), \tau_{NSH})$ embeds into $(\mathcal{P}_n(H), \tau_{HNSH})$ by the mapping
$$Z \mapsto \{\,(T(Z), I(Z), F(Z))\}.$$
Hence, every Neutrosophic superhypertopology can be viewed as a special case of (or is embedded in) a HyperNeutrosophic superhypertopology. □

Theorem 3.8 (SuperHyperNeutrosophic Generalizes HyperNeutrosophic)**.** *Every HyperNeutrosophic superhypertopology is naturally embedded in a SuperHyperNeutrosophic superhypertopology.*

Proof. Let $(\mathcal{P}_n(H), \tau_{HNSH})$ be a HyperNeutrosophic superhypertopological space (Definition 3.1). We construct a SuperHyperNeutrosophic superhypertopological space $(\mathcal{P}_n(H), \tau_{SHNSH})$:

1. *Underlying collection of sets:* Define
$$\tau_{SHNSH} := \tau_{HNSH}.$$
As before, τ_{HNSH} is already a valid superhypertopology, so it satisfies the standard axioms of inclusion, union, and finite intersection.

2. *SuperHyperNeutrosophic assignment:* For each $Y \in \tau_{HNSH}$, we already have a non-empty set $\widetilde{\mu}(Y) \subseteq [0,1]^3$. Define
$$\widetilde{\mu}_n(Y) \;:=\; \{\,\widetilde{\mu}(Y)\} \;\subseteq\; \mathcal{P}_n([0,1]^3),$$
interpreting $\widetilde{\mu}(Y)$ as an element in the first-level power set of $[0,1]^3$. Because the definition of SuperHyperNeutrosophic membership (Definition 3.4) allows for nested sets of membership values, the singleton set $\{\widetilde{\mu}(Y)\}$ trivially satisfies
$$\widetilde{\mu}_n(Y) \;\subseteq\; \mathcal{P}_n([0,1]^3).$$
Indeed, no contradiction arises in placing a single subset $\widetilde{\mu}(Y)$ inside the power set structure $\mathcal{P}_n([0,1]^3)$. The condition $0 \leq T+I+F \leq 3$ is still satisfied for all triples (T,I,F) in $\widetilde{\mu}(Y)$.

Thus, each $Y \in \tau_{SHNSH}$ is assigned a valid *SuperHyperNeutrosophic* membership $\widetilde{\mu}_n(Y)$. Consequently, $(\mathcal{P}_n(H), \tau_{SHNSH})$ is a SuperHyperNeutrosophic superhypertopology, and $(\mathcal{P}_n(H), \tau_{HNSH})$ embeds into it. Therefore, every HyperNeutrosophic superhypertopology can be viewed as a special case of a SuperHyperNeutrosophic superhypertopology. □

Theorem 3.9 (Intersection of HyperNeutrosophic SuperHypertopologies)**.** *Let $\{\tau_\alpha\}_{\alpha\in A}$ be a family of HyperNeutrosophic superhypertopologies on $\mathcal{P}_n(H)$. Then*
$$\tau_\cap \;:=\; \bigcap_{\alpha\in A} \tau_\alpha$$
is also a HyperNeutrosophic superhypertopology on $\mathcal{P}_n(H)$.

Proof. Since each τ_α is a superhypertopology, their intersection $\tau_\cap$ automatically satisfies:

- $\emptyset, \mathcal{P}_n(H) \in \tau_\alpha$ for all α, hence $\emptyset, \mathcal{P}_n(H)$ lie in $\tau_\cap$.
- Closure under arbitrary unions and finite intersections follows by standard arguments for intersections of topologies.

It remains to check the *HyperNeutrosophic assignment*. For every $Z \in \tau_\cap$, we have $Z \in \tau_\alpha$ for all $\alpha \in A$. Thus, each τ_α assigns a non-empty subset $\widetilde{\mu}_\alpha(Z) \subseteq [0,1]^3$ satisfying $0 \leq T+I+F \leq 3$. We can define
$$\widetilde{\mu}_\cap(Z) \;:=\; \bigcap_{\alpha\in A} \widetilde{\mu}_\alpha(Z).$$
While $\widetilde{\mu}_\cap(Z)$ could in principle be empty, one can *refine* or *choose* a canonical assignment to ensure non-emptiness (for instance, by selecting any fixed $\alpha_0 \in A$ and letting $\widetilde{\mu}_\cap(Z) = \widetilde{\mu}_{\alpha_0}(Z)$). Under practical or conventional definitions, one typically chooses a unifying assignment that remains non-empty. Hence $\tau_\cap$ inherits a consistent HyperNeutrosophic assignment, completing the proof. □

Theorem 3.10 (Generation of a HyperNeutrosophic SuperHypertopology by a Subfamily)**.** *Let $\mathcal{B} \subseteq \mathcal{P}(\mathcal{P}_n(H))$ be an arbitrary subfamily. Then there exists a smallest HyperNeutrosophic superhypertopology $\tau_{HNSH}(\mathcal{B})$ that contains $\mathcal{B}$. Furthermore, $\tau_{HNSH}(\mathcal{B})$ can be described as the intersection of all HyperNeutrosophic superhypertopologies containing $\mathcal{B}$.*

Proof. Let
$$\mathscr{S} = \big\{\, \tau \;\big|\; \tau \text{ is a HyperNeutrosophic superhypertopology on } \mathcal{P}_n(H) \text{ with } \mathcal{B} \subseteq \tau \big\}.$$
Note first that $\mathscr{S}$ is nonempty, since the discrete HyperNeutrosophic superhypertopology $\tau_{\max} = \mathcal{P}(\mathcal{P}_n(H))$ contains every subfamily, in particular $\mathcal{B}$.

(i) Existence and minimality. Define
$$\tau_{HNSH}(\mathcal{B}) = \bigcap_{\tau\in\mathscr{S}} \tau.$$

By Theorem 3.9, the intersection of any family of HyperNeutrosophic superhypertopologies is again a HyperNeutrosophic superhypertopology. Hence $\tau_{HNSH}(\mathcal{B})$ is itself a HyperNeutrosophic superhypertopology. Clearly $\tau_{HNSH}(\mathcal{B})$ contains $\mathcal{B}$ (since each $\tau \in \mathscr{S}$ does), and by construction it is contained in every other HyperNeutrosophic superhypertopology that contains $\mathcal{B}$. Thus it is the *smallest* such.

(ii) HyperNeutrosophic assignment. Each $\tau \in \mathscr{S}$ assigns to every $W \in \tau$ a non-empty subset $\widetilde{\mu}_\tau(W) \subseteq [0,1]^3$ satisfying the sum-constraint $0 \leq T + I + F \leq 3$. Since $W \in \tau_{HNSH}(\mathcal{B})$ implies $W \in \tau$ for every $\tau \in \mathscr{S}$, the family $\{\widetilde{\mu}_\tau(W)\}_{\tau \in \mathscr{S}}$ is a nonempty collection of non-empty subsets of $[0,1]^3$. We define

$$\widetilde{\mu}_*(W) = \bigcap_{\tau \in \mathscr{S}} \widetilde{\mu}_\tau(W).$$

Because each $\widetilde{\mu}_\tau(W)$ is nonempty and closed under the constraint $0 \leq T+I+F \leq 3$, their intersection $\widetilde{\mu}_*(W)$ is also nonempty and satisfies the same constraint. Assigning $\widetilde{\mu}_*(W)$ to W endows $\tau_{HNSH}(\mathcal{B})$ with a valid HyperNeutrosophic membership structure.

Hence $\tau_{HNSH}(\mathcal{B})$, together with the assignment $W \mapsto \widetilde{\mu}_*(W)$, is a well-defined HyperNeutrosophic superhypertopology that is minimal among those containing $\mathcal{B}$. □

3.2 Plithogenic SuperHypertopology

A Plithogenic SuperHypertopology extends superhypertopology by incorporating multi-dimensional attributes, contradictions, and appurtenance degrees for subsets within hierarchical structures.

Definition 3.11 (Plithogenic SuperHypertopology)**.** A *Plithogenic SuperHypertopology* on $\mathcal{P}_n(H)$ is a collection τ_{PSH} of subsets of $\mathcal{P}_n(H)$ satisfying:

1. *Superhypertopological Axioms:*
   - $\emptyset \in \tau_{PSH}$ and $\mathcal{P}_n(H) \in \tau_{PSH}$.
   - For any subfamily $\{W_\alpha\}_{\alpha \in I} \subseteq \tau_{PSH}$, we have $\bigcup_{\alpha \in I} W_\alpha \in \tau_{PSH}$.
   - For any finite subfamily $\{W_1, \ldots, W_m\} \subseteq \tau_{PSH}$, we have $\bigcap_{j=1}^{m} W_j \in \tau_{PSH}$.
2. *Plithogenic Assignment:* Each $W \in \tau_{PSH}$ is assigned a *plithogenic structure*

   $$\big(P_W, v_W, Pv_W, pdf_W, pCF_W\big),$$

   where
   - $P_W \subseteq H$ (often we take $P_W = W$ or some sub-collection of W, depending on the application),
   - v_W is an attribute,
   - Pv_W is the set of possible values for v_W,
   - $pdf_W : P_W \times Pv_W \to [0,1]^s$ is the Degree of Appurtenance Function,
   - $pCF_W : Pv_W \times Pv_W \to [0,1]^t$ is the Degree of Contradiction Function,

   all satisfying the axioms in Definition.

The pair $\big(\mathcal{P}_n(H), \tau_{PSH}\big)$ is called a *Plithogenic SuperHypertopological Space*.

Example 3.12 (Plithogenic SuperHypertopology on Sensor-Cluster Activity)**.** Let

$$H = \{\mathrm{S}_1, \mathrm{S}_2\}, \quad \mathcal{P}_1(H) = \{\varnothing, \{\mathrm{S}_1\}, \{\mathrm{S}_2\}, \{\mathrm{S}_1, \mathrm{S}_2\}\}, \quad \mathcal{P}_2(H) = \mathcal{P}(\mathcal{P}_1(H)).$$

Define three "open families'' in $\mathcal{P}_2(H)$:

$$\begin{aligned} W_0 &= \{\varnothing\}, \\ W_1 &= \big\{\{\mathrm{S}_1\}, \{\mathrm{S}_1, \mathrm{S}_2\}\big\}, \\ W_{\max} &= \mathcal{P}_1(H). \end{aligned}$$

One checks easily that $\emptyset, \mathcal{P}_1(H) \in \{W_0, W_1, W_{\max}\}$ and that the collection $\tau_{PSH} = \{W_0, W_1, W_{\max}\}$ is closed under arbitrary unions and finite intersections, so it is a valid superhypertopology.

We now equip each $W \in \tau_{PSH}$ with a plithogenic structure $\big(P_W, v_W, Pv_W, pdf_W, pCF_W\big)$ as follows:

1. For W_0.

$$P_{W_0} = \varnothing, \quad v_{W_0} = \text{"SignalConfidence''}, \quad Pv_{W_0} = \{\mathrm{Low}, \mathrm{High}\}.$$

Since P_{W_0} is empty, we take pdf_{W_0} and pCF_{W_0} to be the unique (empty) mappings; trivially they satisfy the plithogenic axioms.

2. For W_1.

$$P_{W_1} = \{\{\mathrm{S}_1\}, \{\mathrm{S}_1, \mathrm{S}_2\}\}, \quad v_{W_1} = \text{"SignalConfidence''}, \quad Pv_{W_1} = \{\mathrm{Low}, \mathrm{Medium}, \mathrm{High}\}.$$

Define $pdf_{W_1} : P_{W_1} \times Pv_{W_1} \to [0,1]$ by

$$\begin{aligned} &pdf_{W_1}(\{\mathrm{S}_1\}, \mathrm{Low}) = 0.2, \quad pdf_{W_1}(\{\mathrm{S}_1\}, \mathrm{Medium}) = 0.5, \quad pdf_{W_1}(\{\mathrm{S}_1\}, \mathrm{High}) = 0.3, \\ &pdf_{W_1}(\{\mathrm{S}_1, \mathrm{S}_2\}, \mathrm{Low}) = 0.1, \; pdf_{W_1}(\{\mathrm{S}_1, \mathrm{S}_2\}, \mathrm{Medium}) = 0.4, \; pdf_{W_1}(\{\mathrm{S}_1, \mathrm{S}_2\}, \mathrm{High}) = 0.5. \end{aligned}$$

Define $pCF_{W_1} : Pv_{W_1} \times Pv_{W_1} \to [0,1]$ by the matrix

	Low	Medium	High
Low	0	0.3	0.6
Medium	0.3	0	0.7
High	0.6	0.7	0

which is reflexive on the diagonal and symmetric.

3. For $W_{\max}$.

$$P_{W_{\max}} = \mathcal{P}_1(H), \quad v_{W_{\max}} = \text{"SignalConfidence''}, \quad Pv_{W_{\max}} = \{\mathrm{Low}, \mathrm{Medium}, \mathrm{High}\}.$$

Define $pdf_{W_{\max}}$ by

$$\begin{aligned} &pdf_{W_{\max}}(\varnothing, \mathrm{Low}) = 1.0, && pdf_{W_{\max}}(\varnothing, \mathrm{Medium}) = 0.0, && pdf_{W_{\max}}(\varnothing, \mathrm{High}) = 0.0, \\ &pdf_{W_{\max}}(\{\mathrm{S}_1\}, \mathrm{Low}) = 0.5, && pdf_{W_{\max}}(\{\mathrm{S}_1\}, \mathrm{Medium}) = 0.4, && pdf_{W_{\max}}(\{\mathrm{S}_1\}, \mathrm{High}) = 0.1, \\ &pdf_{W_{\max}}(\{\mathrm{S}_2\}, \mathrm{Low}) = 0.4, && pdf_{W_{\max}}(\{\mathrm{S}_2\}, \mathrm{Medium}) = 0.4, && pdf_{W_{\max}}(\{\mathrm{S}_2\}, \mathrm{High}) = 0.2, \\ &pdf_{W_{\max}}(\{\mathrm{S}_1, \mathrm{S}_2\}, \mathrm{Low}) = 0.2, && pdf_{W_{\max}}(\{\mathrm{S}_1, \mathrm{S}_2\}, \mathrm{Medium}) = 0.5, && pdf_{W_{\max}}(\{\mathrm{S}_1, \mathrm{S}_2\}, \mathrm{High}) = 0.3. \end{aligned}$$

Define $pCF_{W_{\max}}$ by

$$pCF_{W_{\max}}(\alpha, \beta) = \begin{cases} 0, & \alpha = \beta, \\ 0.4, & \{\alpha, \beta\} = \{\mathrm{Low}, \mathrm{Medium}\}, \\ 0.6, & \{\alpha, \beta\} = \{\mathrm{Low}, \mathrm{High}\}, \\ 0.5, & \{\alpha, \beta\} = \{\mathrm{Medium}, \mathrm{High}\}. \end{cases}$$

All the above *pdf* values lie in $[0,1]$, each pCF is reflexive and symmetric, and the collection $\tau_{PSH} = \{W_0, W_1, W_{\max}\}$ satisfies the superhypertopological axioms. Therefore, $(\mathcal{P}_2(H), \tau_{PSH})$ with these plithogenic assignments is a concrete *Plithogenic SuperHypertopological Space.*

Theorem 3.13 (Generalization of Neutrosophic SuperHypertopology)**.** *Every Neutrosophic SuperHypertopology τ_{NSH} on $\mathcal{P}_n(H)$ can be embedded into a Plithogenic SuperHypertopology τ_{PSH} on the same $\mathcal{P}_n(H)$.*

Proof. Let $\big(\mathcal{P}_n(H), \tau_{NSH}\big)$ be a Neutrosophic SuperHypertopological Space (as in Definition 2.22). By definition, each $Z \in \tau_{NSH}$ is assigned a neutrosophic triplet

$$\big(T(Z), I(Z), F(Z)\big) \in [0,1]^3,$$

satisfying $0 \leq T(Z) + I(Z) + F(Z) \leq 3$.

We construct a Plithogenic SuperHypertopological Space $\big(\mathcal{P}_n(H), \tau_{PSH}\big)$ as follows:

1. *Same Underlying Set Collection:* Define

   $$\tau_{PSH} := \tau_{NSH}.$$

   Because τ_{NSH} is already a superhypertopology (closed under arbitrary unions and finite intersections, containing $\emptyset$ and $\mathcal{P}_n(H)$), it remains a valid superhypertopology.

2. *Plithogenic Assignment:* For each $Z \in \tau_{PSH}$, we must assign a plithogenic structure

   $$\big(P_Z, v_Z, Pv_Z, pdf_Z, pCF_Z\big).$$

   We define this structure so as to *capture* the existing neutrosophic triple:

   - $P_Z := Z$.
   - v_Z is taken as a *single* attribute (for instance, naming it "NeutrosophicMembership" if desired).
   - $Pv_Z := \{\, T, I, F \,\}$.
   - Define $pdf_Z : Z \times \{T, I, F\} \to [0,1]$ by

     $$pdf_Z(x, T) = T(Z), \quad pdf_Z(x, I) = I(Z), \quad pdf_Z(x, F) = F(Z),$$

     for all $x \in Z$. In other words, we treat $(T(Z), I(Z), F(Z))$ as the same membership degrees for *each* $x \in Z$. Although this might look uniform in x, it is consistent with the fundamental requirement that every pair (p, a) is mapped to a membership degree in $[0,1]^s$. For simplicity, we can take $s = 1$.
   - Define $pCF_Z : \{T, I, F\} \times \{T, I, F\} \to [0,1]$ by

     $$pCF_Z(\alpha, \beta) = 0 \quad \text{for all } \alpha, \beta \in \{T, I, F\}.$$

     This trivial definition satisfies reflexivity and symmetry. (If one prefers, it is possible to incorporate contradictory aspects by letting certain pairs (α, β) have a non-zero contradiction value.)

   The above choices satisfy Definition:

   - $pCF_Z(\alpha, \alpha) = 0$ (reflexivity),
   - $pCF_Z(\alpha, \beta) = pCF_Z(\beta, \alpha)$ (symmetry).

   Since $0 \leq T(Z) + I(Z) + F(Z) \leq 3$, none of the membership degrees exceed 1 in the usual normalization. Hence we have a valid plithogenic set structure for each Z.

Therefore, τ_{PSH} is a *Plithogenic SuperHypertopology* on $\mathcal{P}_n(H)$ (Definition 3.11), and the embedding

$$Z \;\mapsto\; \big(P_Z, v_Z, Pv_Z, pdf_Z, pCF_Z\big)$$

maps each neutrosophic triple to a corresponding plithogenic assignment. This construction proves that every neutrosophic superhypertopology can be realized as a special case within the plithogenic framework. □

Theorem 3.14 (Intersection of Plithogenic SuperHypertopologies)**.** *Let $\{\tau_\alpha\}_{\alpha\in A}$ be a family of plithogenic superhypertopologies on $\mathcal{P}_n(H)$. Then*

$$\tau_\cap := \bigcap_{\alpha\in A} \tau_\alpha$$

is also a plithogenic superhypertopology on $\mathcal{P}_n(H)$.

Proof. Let $\{\tau_\alpha\}_{\alpha\in A}$ be plithogenic superhypertopologies on $\mathcal{P}_n(H)$, and set

$$\tau_\cap = \bigcap_{\alpha\in A} \tau_\alpha.$$

We check the two parts of the definition.

1. Superhypertopology axioms.

(a) *Empty and universal sets.* Since each τ_α contains $\varnothing$ and $\mathcal{P}_n(H)$, their intersection $\tau_\cap$ also contains these two sets.

(b) *Closure under arbitrary unions.* Let $\{W_i\}_{i\in I} \subseteq \tau_\cap$. Then for each α, $W_i \in \tau_\alpha$ for all i, and since τ_α is a superhypertopology,

$$\bigcup_{i\in I} W_i \in \tau_\alpha.$$

This holds for every α, so $\bigcup_{i\in I} W_i$ lies in the intersection $\tau_\cap$.

(c) *Closure under finite intersections.* Similarly, if $W_1, \ldots, W_m \in \tau_\cap$, then for each α, $\bigcap_{j=1}^m W_j \in \tau_\alpha$, whence $\bigcap_{j=1}^m W_j \in \tau_\cap$.

2. Plithogenic assignment. We must assign to each $W \in \tau_\cap$ a valid plithogenic structure

$$\big(P_W, v_W, Pv_W, pdf_W, pCF_W\big)$$

. Since $W \in \tau_\cap$, it belongs to *every* τ_α. In particular, pick an arbitrary index $\alpha_0 \in A$. Then τ_{α_0} already provides

$$\big(P_W^{(\alpha_0)},\ v_W^{(\alpha_0)},\ Pv_W^{(\alpha_0)},\ pdf_W^{(\alpha_0)},\ pCF_W^{(\alpha_0)}\big).$$

Define

$$P_W := P_W^{(\alpha_0)}, \quad v_W := v_W^{(\alpha_0)}, \quad Pv_W := Pv_W^{(\alpha_0)}, \quad pdf_W := pdf_W^{(\alpha_0)}, \quad pCF_W := pCF_W^{(\alpha_0)}.$$

Because τ_{α_0} is plithogenic, this tuple satisfies:

- $P_W \subseteq H$ and v_W, Pv_W are as required,
- $pdf_W : P_W \times Pv_W \to [0,1]^s$ is well-defined,
- $pCF_W : Pv_W \times Pv_W \to [0,1]^t$ is reflexive and symmetric.

Thus each $W \in \tau_\cap$ receives a valid plithogenic assignment. Hence $\tau_\cap$ is a plithogenic superhypertopology. □

Theorem 3.15 (Generation of a Plithogenic SuperHypertopology from a Subfamily)**.** *Let $\mathcal{B} \subseteq \mathcal{P}(\mathcal{P}_n(H))$ be an arbitrary subfamily. There exists a smallest plithogenic superhypertopology $\tau_{PSH}(\mathcal{B})$ on $\mathcal{P}_n(H)$ that contains $\mathcal{B}$. Moreover,*

$$\tau_{PSH}(\mathcal{B}) \ = \ \bigcap\Big\{\tau \,\Big|\, \tau \text{ is a plithogenic superhypertopology with } \mathcal{B} \subseteq \tau\Big\}.$$

Proof. Let $\mathcal{B} \subseteq \mathcal{P}(\mathcal{P}_n(H))$. Consider the collection

$$\mathscr{S} = \big\{\tau \;\big|\; \tau \text{ is a plithogenic superhypertopology on } \mathcal{P}_n(H),\ \mathcal{B} \subseteq \tau\big\}.$$

Note that $\mathscr{S}$ is nonempty (since $\mathcal{P}(\mathcal{P}_n(H))$ itself is a trivial plithogenic superhypertopology containing $\mathcal{B}$). Define

$$\tau_{PSH}(\mathcal{B}) = \bigcap_{\tau \in \mathscr{S}} \tau.$$

By Theorem 3.14, the intersection of any family of plithogenic superhypertopologies is again a plithogenic superhypertopology. Moreover:

- $\tau_{PSH}(\mathcal{B})$ contains $\mathcal{B}$ because each $\tau \in \mathscr{S}$ does.
- It is *smallest* with this property: if τ' is any plithogenic superhypertopology containing $\mathcal{B}$, then $\tau' \in \mathscr{S}$, so $\tau_{PSH}(\mathcal{B}) \subseteq \tau'$.

Finally, to assign plithogenic structures on $\tau_{PSH}(\mathcal{B})$: for each $W \in \tau_{PSH}(\mathcal{B})$, since $W \in \tau$ for every $\tau \in \mathscr{S}$, pick any one $\tau_0 \in \mathscr{S}$ and use its assignment for W. This yields a well-defined plithogenic structure on $\tau_{PSH}(\mathcal{B})$. Thus $\tau_{PSH}(\mathcal{B})$ is the desired smallest plithogenic superhypertopology containing $\mathcal{B}$. □

4 Conclusion

In this paper, we introduced two novel structures—*Plithogenic Superhypertopology* and *Superhyper-Neutrosophic Superhypertopology*—on $\mathcal{P}_n(H)$. We provided their formal definitions, demonstrated how they generalize existing topological and uncertain-set frameworks, and analyzed their key theoretical properties. In the future, we aim to further investigate the mathematical characteristics of these concepts and explore possible extensions employing HyperPlithogenic Sets [53], Soft Sets [54,55], Bipolar Neutrosophic Sets [56, 57], and Topological HyperGraphs [58].

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Consent to Participate

Not applicable.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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Chapter 9

HyperFuzzy Graph Neural Networks and HyperPlithogenic Graph Neural Networks: Theoretical Foundations

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Abstract

Graph Neural Networks (GNNs) have become an influential framework for learning and inference on graph-structured data [1]. In the realm of uncertainty modeling, Fuzzy Graphs and Neutrosophic Graphs are increasingly recognized as essential tools [2]. Plithogenic Graphs, in particular, offer a broader generalization, encompassing both Fuzzy and Neutrosophic Graphs [3]. Building upon these foundations, recent advancements have introduced the Fuzzy Graph Neural Network, Neutrosophic Graph Neural Network, and Plithogenic Graph Neural Network [4].

In this paper, we further expand these concepts by introducing HyperFuzzy Graphs, HyperNeutrosophic Graphs, and HyperPlithogenic Graphs, which generalize Fuzzy Graphs, Neutrosophic Graphs, and Plithogenic Graphs, respectively. We also define their corresponding neural network architectures: HyperFuzzy Graph Neural Networks, HyperNeutrosophic Graph Neural Networks, and HyperPlithogenic Graph Neural Networks, and provide a detailed analysis of their theoretical properties. As an extension, we explore SuperHyperFuzzy Graph Neural Networks, SuperHyperNeutrosophic Graph Neural Networks, and SuperHyperPlithogenic Graph Neural Networks. Although the focus of this paper is primarily theoretical, we expect that future research will explore the practical implementations and applications of these advanced models.

Keywords: Neutrosophic Graph, Fuzzy Graph, Graph Neural Network, Plithogenic Graph, Hyperstructure

1 Introduction

1.1 Graph Neural Networks

Recent advances in machine learning [5], artificial intelligence [6], deep learning [7], natural language processing [8], and big data analytics [9] have accelerated innovation across many domains. At the heart of these developments are neural networks, which provide flexible tools for tackling complex computational tasks. A neural network is a computational model inspired by biological neural systems and is widely used for pattern recognition, classification, and prediction [10]. Building on this foundation, *Graph Neural Networks* (GNNs) extend learning to graph-structured data, explicitly leveraging nodes, edges, and their features to encode relational structure [1, 11]. From a mathematical perspective, graph theory supplies the formal language for these models by analyzing networks of vertices and edges and their connectivity properties [12].

Hypergraph Neural Networks (HGNNs) generalize GNNs by introducing hyperedges that capture higher-order relations among multiple nodes, enabling the representation of more complex interac-tions [13–15]. In a complementary direction, *Hypernetworks* produce network parameters dynami-cally and have been extensively explored for their adaptability and expressiveness [16, 17]. Beyond undirected settings, *Directed Graph Neural Networks* model asymmetric dependencies in directed graphs [18, 19], while *Mixed Graph Neural Networks* address hybrid structures containing both di-rected and undirected edges [20].

1.2 Uncertain Graphs

Zadeh's fuzzy sets [21] transformed the mathematical treatment of uncertainty and have been applied broadly in graph theory [22, 23], algebra [24], topology [25], and logic [26], providing a principled way to manage imprecision in real systems (cf. [27, 28]). To accommodate more intricate forms of

uncertainty, *neutrosophic sets* were proposed [29,30] and have since been studied in graph theory [31], decision making [32], algebra [33], topology [34], and logic [29]. Other influential frameworks include *Intuitionistic Fuzzy Sets* [35], *Soft Sets* [36], *Rough Sets* [37,38], and *Vague Sets* [39,40]. Among these, *Plithogenic Sets* are notable for unifying multiple uncertainty paradigms and for handling diverse, potentially conflicting attributes within a single model.

This progression has also inspired higher-order generalizations such as *Hyperfuzzy* [41,42], *Hyperneutrosophic* [43,44], and *Hyperplithogenic* sets [43]. Further, *SuperHyperfuzzy*, *SuperHyperneutrosophic*, and *SuperHyperplithogenic* models [45, 46] incorporate hierarchical structures based on the n-fold powerset to better capture multi-level uncertainty. In this paper we investigate uncertain *graph* models—including Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Graphs—that extend classical graph theory by embedding uncertainty into vertices, edges, or their relationships. These models support nuanced analyses of ambiguous interactions and are increasingly relevant in real-world applications.

Uncertain graphs play a growing role in machine learning, AI, and NLP (cf. [47–49]). Recent developments such as *Fuzzy Graph Neural Networks* and *Neutrosophic Graph Neural Networks* leverage these formalisms—along with their intuitionistic and plithogenic counterparts—to address learning tasks on complex, uncertain data [4].

1.3 Our Contribution

We advance this line of work by introducing and studying three hyper-level graph models—*HyperFuzzy Graphs*, *HyperNeutrosophic Graphs*, and *HyperPlithogenic Graphs*—which generalize Fuzzy, Neutrosophic, and Plithogenic Graphs, respectively. We also define their neural analogues: *HyperFuzzy Graph Neural Networks*, *HyperNeutrosophic Graph Neural Networks*, and *HyperPlithogenic Graph Neural Networks*. Our focus is theoretical: we formalize these objects and analyze their principal properties, while leaving practical implementations and applications to future work.

2 Preliminaries and Definitions

In this section, we provide a brief overview of the definitions and notations used throughout this paper. While we aim to make the content accessible to readers from various backgrounds, it is not possible to cover all relevant details comprehensively. Readers are encouraged to consult the referenced literature for additional information as needed.

2.1 Basic Graph Neural Network Concepts

Here are several definitions of Graph Neural Networks (GNNs). Readers may refer to the lecture notes or the introduction for further details(cf. [50, 51]).

Definition 2.1 (Graph Neural Network (GNN))**.** (cf. [52]) Let $G = (V, E)$ be a graph, where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices and $E \subseteq V \times V$ is the set of edges. Each vertex $v_i \in V$ is associated with a feature vector $\mathbf{x}_i \in \mathbb{R}^d$, and each edge $(v_i, v_j) \in E$ may optionally have a feature $\mathbf{e}_{ij} \in \mathbb{R}^k$.

A Graph Neural Network (GNN) computes node representations $\mathbf{h}_i^{(t)} \in \mathbb{R}^d$ at each layer t, using the graph structure and associated features.

Definition 2.2 (Key Components of Graph Neural Network)**.** (cf. [52]) Several key components of Graph Neural Networks are outlined below.

1. Node Initialization: At the initial layer ($t = 0$), the node representations are initialized as:

$$\mathbf{h}_i^{(0)} = \mathbf{x}_i, \quad \forall v_i \in V.$$

2. Message Passing(cf. [53]): At each layer t, messages are exchanged between connected nodes. The messages received by a node v_i from its neighbors are computed as:

$$\mathbf{m}_i^{(t+1)} = \sum_{v_j \in \mathcal{N}(i)} \varphi_m(\mathbf{h}_i^{(t)}, \mathbf{h}_j^{(t)}, \mathbf{e}_{ij}),$$

where:

- $\mathcal{N}(i)$ is the set of neighbors of v_i,
- $\varphi_m : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^d$ is the message function.

3. Node Update: (cf. [54]) The representation of each node is updated using the received messages:

$$\mathbf{h}_i^{(t+1)} = \varphi_u(\mathbf{h}_i^{(t)}, \mathbf{m}_i^{(t+1)}),$$

where $\varphi_u : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is the update function.

4. Readout Function: For graph-level tasks, a global representation $\mathbf{z}_G$ is computed by aggregating node representations:

$$\mathbf{z}_G = \varphi_r\left(\{\mathbf{h}_i^{(T)} \mid v_i \in V\}\right),$$

where φ_r is the readout function (e.g., summation, averaging, or max-pooling).

2.2 Uncertain Graph

The concept of the Fuzzy Set, introduced approximately half a century ago, has spurred the development of various graph theories aimed at modeling uncertainty [21]. In this section, we outline definitions for several frameworks, including Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Single-Valued Pentapartitioned Neutrosophic Graphs.

A Fuzzy Graph is frequently analyzed in the context of a Crisp Graph [55]. To provide a foundation, we begin by presenting the definition of a Crisp Graph [55].

Definition 2.3 (Crisp Graph)**.** (cf. [55]) A *Crisp Graph* $G = (V, E)$ is defined as follows:

1. V: A non-empty finite set of vertices (or nodes).
2. $E \subseteq \{\{u, v\} \mid u, v \in V \text{ and } u \neq v\}$: A set of unordered pairs of vertices, called edges. Each edge is associated with exactly two vertices, referred to as its endpoints. An edge is said to connect its endpoints.

Special Cases

- A graph G with $E = \varnothing$ is called an *edgeless graph*.

We now introduce the concepts of Fuzzy Graph, Intuitionistic Fuzzy Graph, and Neutrosophic Graph. For more detailed information, readers may refer to survey papers such as [55].

Definition 2.4 (Unified Framework for Uncertain Graphs)**.** (cf. [56]) Let $G = (V, E)$ be a classical graph, where V is the set of vertices and E is the set of edges. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is associated with membership values to represent various degrees of truth, indeterminacy, falsity, and other measures of uncertainty.

1. *Fuzzy Graph* (cf. [57])

- Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0,1]$.
- Each edge $e = (u,v) \in E$ is assigned a membership degree $\mu(u,v) \in [0,1]$.

2. *Intuitionistic Fuzzy Graph (IFG)* (cf. [58, 59])
   - Each vertex $v \in V$ has two values: $\mu_A(v) \in [0,1]$ (degree of membership) and $\nu_A(v) \in [0,1]$ (degree of non-membership), satisfying $\mu_A(v) + \nu_A(v) \leqslant 1$.
   - Each edge $e = (u,v) \in E$ has two values: $\mu_B(u,v) \in [0,1]$ and $\nu_B(u,v) \in [0,1]$, with $\mu_B(u,v) + \nu_B(u,v) \leqslant 1$.
3. *Neutrosophic Graph* (cf. [60])
   - Each vertex $v \in V$ is associated with a triplet
   $$\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$$
   , where
   $$\sigma_T(v), \sigma_I(v), \sigma_F(v) \in [0,1]$$
   and $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leqslant 3$.
   - Each edge $e = (u,v) \in E$ is associated with a triplet $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$.

A Plithogenic Graph is a generalized graph based on the concept of a Plithogenic Set (cf. [61]). This graph is known for its ability to generalize structures such as Fuzzy Graphs and Neutrosophic Graphs described earlier. The definition is provided below [3].

Definition 2.5 (Plithogenic Graph)**.** [62, 63] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph* PG is defined as:

$$PG = (PM, PN)$$

where:

1. *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:
   - $M \subseteq V$ is the set of vertices.
   - l is an attribute associated with the vertices.
   - Ml is the range of possible attribute values.
   - $adf : M \times Ml \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
   - $aCf : Ml \times Ml \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.
2. *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:
   - $N \subseteq E$ is the set of edges.
   - m is an attribute associated with the edges.
   - Nm is the range of possible attribute values.
   - $bdf : N \times Nm \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.
   - $bCf : Nm \times Nm \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

1. *Edge Appurtenance Constraint*: For all $(x,a), (y,b) \in M \times Ml$:
   $$bdf\,((xy),(a,b)) \leqslant \min\{adf(x,a), adf(y,b)\}$$
   where $xy \in N$ is an edge between vertices x and y, and $(a,b) \in Nm \times Nm$ are the corresponding attribute values.

2. *Contradiction Function Constraint*: For all $(a,b),(c,d) \in Nm \times Nm$:

$$bCf\left((a,b),(c,d)\right) \leqslant \min\{aCf(a,c), aCf(b,d)\}$$

3. *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned} aCf(a,a) &= 0, && \forall a \in Ml \\ aCf(a,b) &= aCf(b,a), && \forall a,b \in Ml \\ bCf(a,a) &= 0, && \forall a \in Nm \\ bCf(a,b) &= bCf(b,a), && \forall a,b \in Nm \end{aligned}$$

Example 2.6 (Plithogenic Graph of Inter-Department Communication)**.** Let $G = (V,E)$ be the graph of three employees and their direct communication links:

$$V = \{A,B,C\}, \quad E = \{\,(A,B),\,(B,C)\}.$$

We model each employee's *department affiliation* and each communication link's *department-pair context* as plithogenic attributes.

1. Plithogenic Vertex Set PM.

- $M = V = \{A,B,C\}$.
- Vertex attribute $l =$ "department."
- Value set $Ml = \{\mathrm{HR},\mathrm{IT}\}$.
- Degree of Appurtenance Function

$$adf : M \times Ml \to [0,1],$$

giving each employee X a degree of membership in each department:

$$\begin{aligned} adf(A,\mathrm{HR}) &= 0.7, \quad adf(A,\mathrm{IT}) = 0.3, \\ adf(B,\mathrm{HR}) &= 0.2, \quad adf(B,\mathrm{IT}) = 0.8, \\ adf(C,\mathrm{HR}) &= 0.5, \quad adf(C,\mathrm{IT}) = 0.5. \end{aligned}$$

Here $adf(A,\mathrm{HR}) = 0.7$ means 70% evidence that A belongs to HR.

- Degree of Contradiction Function

$$aCf : Ml \times Ml \to [0,1],$$

measuring how "incompatible" two department labels are:

$$aCf(\mathrm{HR},\mathrm{HR}) = 0, \quad aCf(\mathrm{IT},\mathrm{IT}) = 0, \quad aCf(\mathrm{HR},\mathrm{IT}) = aCf(\mathrm{IT},\mathrm{HR}) = 0.9.$$

2. Plithogenic Edge Set PN.

We take each edge's context to be the ordered pair of its endpoints' departments, so

$$N = E, \quad m = \text{"department pair"}, \quad Nm = Ml \times Ml = \{(\mathrm{HR},\mathrm{HR}),(\mathrm{HR},\mathrm{IT}),(\mathrm{IT},\mathrm{HR}),(\mathrm{IT},\mathrm{IT})\}.$$

- Edge-DAF

$$bdf : N \times Nm \to [0,1],$$

bound by the "min" of the two endpoint's adf values:

$$\begin{aligned} bdf\big((A,B),(\mathrm{HR},\mathrm{IT})\big) = 0.6 &\leqslant \min\{adf(A,\mathrm{HR}), adf(B,\mathrm{IT})\} = 0.7,\\ bdf\big((A,B),(\mathrm{IT},\mathrm{HR})\big) = 0.2 &\leqslant \min\{0.3, 0.2\} = 0.2,\\ bdf\big((A,B),(\mathrm{HR},\mathrm{HR})\big) = 0.2 &\leqslant \min\{0.7, 0.2\} = 0.2,\\ bdf\big((A,B),(\mathrm{IT},\mathrm{IT})\big) = 0.3 &\leqslant \min\{0.3, 0.8\} = 0.3, \end{aligned}$$

and similarly for edge (B,C):

$$\begin{aligned} bdf\big((B,C),(\mathrm{HR},\mathrm{IT})\big) = 0.2 &\leqslant \min\{0.2, 0.5\} = 0.2,\\ bdf\big((B,C),(\mathrm{IT},\mathrm{HR})\big) = 0.5 &\leqslant \min\{0.8, 0.5\} = 0.5,\\ bdf\big((B,C),(\mathrm{HR},\mathrm{HR})\big) = 0.1 &\leqslant \min\{0.2, 0.5\} = 0.2,\\ bdf\big((B,C),(\mathrm{IT},\mathrm{IT})\big) = 0.4 &\leqslant \min\{0.8, 0.5\} = 0.5. \end{aligned}$$

These values express how well each edge's communication matches the specified department pairing.

- Edge-DCF
$$bCf : Nm \times Nm \to [0,1],$$
measuring contradiction between two department pairs:
$$\begin{aligned} &bCf\big((\mathrm{HR},\mathrm{IT}),(\mathrm{IT},\mathrm{HR})\big) = 0.8,\\ &bCf\big((\mathrm{HR},\mathrm{HR}),(\mathrm{IT},\mathrm{IT})\big) = 0.7,\\ &bCf\big((a,a),(a,b)\big) = 0, \quad bCf\big((a,a),(b,a)\big) = 0, \end{aligned}$$
and symmetric in its two arguments. Every bCf value respects
$$bCf\big((a_1,a_2),(b_1,b_2)\big) \leqslant \min\{\, aCf(a_1,b_1),\, aCf(a_2,b_2)\},$$
e.g. $0.8 \leqslant \min\{aCf(\mathrm{HR},\mathrm{IT}),\, aCf(\mathrm{IT},\mathrm{HR})\} = 0.9$.
- *Reflexivity and symmetry:* both aCf and bCf satisfy
$$aCf(x,x) = 0, \quad aCf(x,y) = aCf(y,x), \quad bCf(e,e) = 0, \quad bCf(e_1,e_2) = bCf(e_2,e_1).$$

Interpretation.

- The vertex–DAF $adf(A,\mathrm{HR}) = 0.7$ means "employee A is 70% HR,"
- The edge–DAF $bdf((A,B),(\mathrm{HR},\mathrm{IT})) = 0.6$ means "60% of $A-B$ messages occur under an HR→IT context,"
- The vertex–DCF $aCf(\mathrm{HR},\mathrm{IT}) = 0.9$ indicates "HR vs IT are highly contradictory roles,"
- The edge–DCF $bCf((\mathrm{HR},\mathrm{IT}),(\mathrm{IT},\mathrm{HR})) = 0.8$ indicates "forward vs reverse department-pair contexts are strongly conflicting."

Thus (PM, PN) defines a *Plithogenic Graph* modeling both the degrees and contradictions of departmental affiliations and their communications.

2.3 Fuzzy Graph Neural Network (F-GNN)

In this subsection, we introduce the concept of the Fuzzy Graph Neural Network (F-GNN). A Fuzzy Graph Neural Network (F-GNN) is a graph inference model that combines the principles of fuzzy logic and graph neural networks (GNNs). It is specifically designed to address fuzzy and uncertain data within graph-structured information (cf. [64, 65]). Below, we present the formal definition of F-GNN.

Definition 2.7 (Fuzzification Layer)**.** Let $G = (V, E)$ be a graph with node features $X_V \in \mathrm{R}^{|V| \times d_v}$ and edge features $X_E \in \mathrm{R}^{|E| \times d_e}$. Fix $M \in \mathbb{N}$. A *fuzzification layer* is specified by

$$\{\mu_j^V : \mathrm{R}^{d_v} \to [0,1]\}_{j=1}^M, \quad \{\mu_j^E : \mathrm{R}^{d_e} \to [0,1]\}_{j=1}^M,$$

where each μ_j^V (resp. μ_j^E) is a membership function (e.g. a Gaussian or sigmoid) that assigns to node v (resp. edge e) an M-dimensional fuzzy membership vector

$$\mathcal{F}_V(v) = \big[\mu_1^V(x_v), \ldots, \mu_M^V(x_v)\big], \quad \mathcal{F}_E(e) = \big[\mu_1^E(x_e), \ldots, \mu_M^E(x_e)\big].$$

Definition 2.8 (Fuzzy Rule Layer)**.** Choose a collection of K *fuzzy inference rules*. Each rule k is specified by indices $(i_k, j_k) \in \{1, \ldots, M\}^2$ and a weight matrix $W_k \in \mathrm{R}^{d \times 2d}$. Given hidden states $h_v, h_u \in \mathrm{R}^d$ and an edge feature $h_{uv} \in \mathrm{R}^d$, the *firing strength* of rule k on edge (u, v) is

$$r_k(u, v) = \mathrm{T}\big(\mu_{i_k}^V(h_v),\, \mu_{j_k}^E(h_{uv})\big),$$

where $\mathrm{T} : [0,1]^2 \to [0,1]$ is a t-norm (e.g. $\mathrm{T}(a, b) = a\, b$). The *rule output message* from u to v is

$$m_{u \to v}^{(k)} = W_k \left[h_u \,\|\, h_v\right] \in \mathrm{R}^d.$$

Definition 2.9 (Defuzzification and Layer Update)**.** At each layer ℓ, suppose node v has current feature $h_v^{(\ell-1)}$ and each edge (u, v) has feature $h_{uv}^{(\ell-1)}$. Then

$$m_v^{(\ell)} = \sum_{u \in \mathcal{N}(v)} \sum_{k=1}^K r_k^{(\ell)}(u, v)\, m_{u \to v}^{(k)}, \quad r_k^{(\ell)}(u, v) = \mathrm{T}\big(\mu_{i_k}^V(h_v^{(\ell-1)}),\, \mu_{j_k}^E(h_{uv}^{(\ell-1)})\big).$$

The new node feature is

$$h_v^{(\ell)} = \sigma\big(m_v^{(\ell)} + h_v^{(\ell-1)}\big),$$

where σ is a nonlinear activation (e.g. ReLU). Stacking L such layers yields final embeddings $H^{(L)} = \{h_v^{(L)}\}_{v \in V}$. A readout head (e.g. Softmax) produces predictions from $H^{(L)}$.

Example 2.10 (F-GNN for Fuzzy Friend Recommendation)**.** Let G be a social graph with users V and edges E. Each user v has profile $x_v \in \mathrm{R}^d$; each edge (u, v) has interaction vector $x_{uv} \in \mathrm{R}^d$.

1. Fuzzification: Define three node fuzzifiers for "young," "mid," "senior" profiles:

$$\mu_j^V(x_v) = \exp\big(-\|x_v - c_j\|^2/\sigma_j^2\big), \quad j = 1, 2, 3,$$

and two edge fuzzifiers for "frequent chat" and "shared groups":

$$\mu_1^E(x_{uv}) = \tfrac{\mathrm{msgs}_{uv}}{\max \mathrm{msgs}}, \quad \mu_2^E(x_{uv}) = \tfrac{\mathrm{groups}_{uv}}{\max \mathrm{groups}}.$$

2. Rules: Two example rules:

Rule 1: IF v is young AND (u, v) is frequentChat THEN $m_{u \to v}^{(1)} = W_1[h_u \| h_v]$,
Rule 2: IF v is mid AND (u, v) is sharedGroups THEN $m_{u \to v}^{(2)} = W_2[h_u \| h_v]$.

3. Defuzzification/Update: Compute

$$r_k(u, v) = \mu_{i_k}^V(h_v)\, \mu_{j_k}^E(h_{uv}), \quad m_v = \sum_{u,k} r_k(u, v)\, m_{u \to v}^{(k)}, \quad h_v' = \mathrm{ReLU}(m_v + h_v).$$

After L layers, apply Softmax to $h_v^{(L)}$ to rank friend suggestions. The fuzzy memberships μ_j^V, μ_j^E capture uncertainty in interests and interactions, yielding more nuanced recommendations than a binary GNN.

2.4 Neutrosophic Graph Neural Network (N-GNN)

In this subsection, we define the concept of the *Neutrosophic Graph Neural Network (N-GNN)* and demonstrate how it generalizes the Fuzzy Graph Neural Network (F-GNN) [4, 66].

Definition 2.11 (Neutrosophic Graph Neural Network (N-GNN))**.** [4] A Neutrosophic Graph Neural Network (N-GNN) is a graph inference model that incorporates neutrosophic logic into graph neural networks to handle uncertain, indeterminate, and inconsistent data. Formally, an N-GNN is defined as:

$$\text{N-GNN} = (G, \mathcal{N}_V, \mathcal{N}_E, \mathcal{R}_N, \mathcal{D}_N),$$

where:

- $G = (V, E)$ is a graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges.
- $\mathcal{N}_V : \mathcal{X}_V \to [0,1]^3$ is the neutrosophic fuzzification function for vertices.
- $\mathcal{N}_E : \mathcal{X}_E \to [0,1]^3$ is the neutrosophic fuzzification function for edges.
- $\mathcal{R}_N$ encodes neutrosophic rules for information aggregation.
- $\mathcal{D}_N : \mathcal{Y}_N \to \mathbb{R}$ is the defuzzification function.

Definition 2.12 (Operations in N-GNN)**.** Given a graph $G = (V, E)$ with vertex features X_V and edge features X_E, the N-GNN operates as follows:

1. **Neutrosophic Fuzzification Layer:** Each vertex $v \in V$ and edge $e \in E$ is mapped to neutrosophic membership triplets:
$$\mathcal{N}_V(v) = (\mu_T(v), \mu_I(v), \mu_F(v)), \quad \mathcal{N}_E(e) = (\mu_T(e), \mu_I(e), \mu_F(e)).$$
2. **Rule Layer:** Neutrosophic rules aggregate information:
$$\text{IF } v \text{ has } (\mu_T^v, \mu_I^v, \mu_F^v) \text{ AND } u \text{ has } (\mu_T^u, \mu_I^u, \mu_F^u) \text{ THEN } y_k = f_k\left(\mathcal{N}_V(v), \mathcal{N}_V(u)\right),$$
where f_k is a trainable function.
3. **Normalization Layer:** The firing strengths r_k are normalized:
$$r_k = \text{Comb}\left(\mathcal{N}_V(v), \mathcal{N}_V(u)\right), \quad \hat{r}_k = \frac{r_k}{\sum_{j=1}^{K} r_j}.$$
4. **Defuzzification Layer:** Aggregated outputs produce crisp predictions:
$$y = \sum_{k=1}^{K} \hat{r}_k \cdot f_k(x_v, x_u).$$

Definition 2.13 (Multi-layer N-GNN)**.** For a multi-layer N-GNN, the output of the l-th layer is:

$$H^{(l)} = \sigma\left(f_\theta^{(l)}\left(H^{(l-1)}, A\right) + H^{(l-1)}\right),$$

where:

- $H^{(l)}$ is the feature matrix at layer l,
- A is the adjacency matrix,
- $f_\theta^{(l)}$ is a trainable function,
- σ is a non-linear activation function.

The final output is:

$$Y = \text{Softmax}\left(H^{(L)}\right),$$

where L is the total number of layers.

Example 2.14 (Neutrosophic Social Network)**.** Consider a small social network represented by a graph $G = (V, E)$, where each vertex in $V = \{\text{Alice}, \text{Bob}, \text{Carol}\}$ denotes a person, and edges in E indicate potential communication links. Suppose we have uncertain or incomplete data on how each person supports a given topic (e.g., a proposed policy). We assign each vertex $v \in V$ a neutrosophic triplet

$$\big(\mu_T(v),\ \mu_I(v),\ \mu_F(v)\big),$$

interpreted as:

- $\mu_T(v) \in [0, 1]$: Degree of support (*truth*),
- $\mu_I(v) \in [0, 1]$: Degree of uncertainty (*indeterminacy*),
- $\mu_F(v) \in [0, 1]$: Degree of opposition (*falsity*),

subject to $0 \leqslant \mu_T(v) + \mu_I(v) + \mu_F(v) \leqslant 3$. For instance:

$$\mu_T(\text{Alice}) = 0.8, \quad \mu_I(\text{Alice}) = 0.1, \quad \mu_F(\text{Alice}) = 0.1,$$

indicating Alice is fairly supportive and only slightly uncertain or opposed. Bob might have

$$(0.5, 0.4, 0.1),$$

reflecting moderate support with higher uncertainty, and Carol could be

$$(0.2, 0.3, 0.5),$$

indicating partial support but more opposition.

An **N-GNN** processes these neutrosophic memberships across edges. If $(\text{Alice}, \text{Bob}) \in E$, we also assign edge-based triplets capturing their communication reliability or shared interests:

$$\mathcal{N}_E(\text{Alice}, \text{Bob}) \ = \ (0.7,\ 0.2,\ 0.1).$$

During the *Rule Layer*, the model combines vertex triplets (e.g., $\mu_T(\text{Alice}), \ldots$) and edge triplets $(\mu_T(\text{Alice}, \text{Bob}), \ldots)$ via neutrosophic rules to infer how strongly Alice's stance might influence Bob. If both $\mu_T(\text{Alice})$ and $\mu_T(\text{Alice}, \text{Bob})$ are high, Bob's next state might shift toward increased support.

By *Defuzzification*, the model outputs a crisp prediction for each vertex—e.g., a final "level of agreement" or "likelihood of adopting the policy." This small example highlights how uncertain, indeterminate, and contradictory stances within a social network can be modeled in an N-GNN to produce more nuanced predictions than standard graph neural networks.

Example 2.15 (Industrial Sensor Network with Neutrosophic Readings)**.** Suppose a factory floor is monitored by sensors measuring temperature, vibration, and chemical concentration. Let $G = (V, E)$ represent the network, where each vertex $v \in V$ is a sensor node and edges indicate direct communication paths. Due to noise or device calibration issues, each sensor's readings may be partially *true* (μ_T), partially *indeterminate* (μ_I) if the reading is unreliable or ambiguous, and partially *false* (μ_F) if the sensor likely gave a spurious measurement.

For sensor v, define

$$\mathcal{N}_V(v) \ = \ \big(\mu_T(v),\ \mu_I(v),\ \mu_F(v)\big),$$

reflecting the confidence in v's reported data. If a sensor reading is stable and well-calibrated, $\mu_T(v) \approx 1$. If the sensor is malfunctioning or affected by unknown factors, $\mu_I(v)$ may be large, and $\mu_F(v)$ might capture known error patterns. Edges can also carry neutrosophic information (e.g., (T, I, F) describing the reliability of data transmission).

In an N-GNN, these triplets flow into a *Rule Layer* that combines neighbor sensor data:

$$\text{IF } \mu_T(v) \text{ is high AND } \mu_T(u) \text{ is moderate THEN } y_k = \dots,$$

where $\hat{r}_k$ normalizes the rule's firing strength. Finally, the *Defuzzification Layer* aggregates these outputs into a predictive score, such as "probability of abnormal temperature" or "likelihood of machine fault." This approach effectively manages uncertainty (due to sensor errors), indeterminacy (unexplained or partial readings), and falsehood (recognized sensor faults) within the same graph neural network framework.

2.5 Plithogenic Graph Neural Network (P-GNN)

A Plithogenic Graph Neural Network (P-GNN) integrates plithogenic logic into graph neural networks, handling data with degrees of appurtenance and contradiction [4].

Definition 2.16 (Plithogenic Graph Neural Network (P-GNN))**.** [4] A Plithogenic Graph Neural Network (P-GNN) integrates plithogenic logic into graph neural networks to handle graph-structured data with varying degrees of appurtenance and contradiction. Formally, a P-GNN is defined as:

$$\text{P-GNN} = (G, \mathcal{P}_V, \mathcal{P}_E, \mathcal{R}_P, \mathcal{D}_P),$$

where:

- $G = (V, E)$ is a graph, where V is the set of vertices, and $E \subseteq V \times V$ is the set of edges.
- $\mathcal{P}_V : \mathcal{X}_V \to [0,1]^s$ is the plithogenic fuzzification function for vertices.
- $\mathcal{P}_E : \mathcal{X}_E \to [0,1]^s$ is the plithogenic fuzzification function for edges.
- $\mathcal{R}_P$ encodes plithogenic rules for information aggregation.
- $\mathcal{D}_P : \mathcal{Y}_P \to \mathbb{R}$ is the plithogenic defuzzification function.

Definition 2.17 (Operations in P-GNN)**.** Given an input graph $G = (V, E)$ with vertex features X_V and edge features X_E, the P-GNN operates as follows:

1. **Plithogenic Fuzzification Layer:** Each vertex $v \in V$ and edge $e \in E$ is fuzzified into plithogenic membership values using degrees of appurtenance (DAF) and contradiction (DCF).
2. **Rule Layer:** Plithogenic rules aggregate information. For example:

   $$\text{IF } v \text{ has DAF } \alpha_v \text{ AND } u \text{ has DAF } \alpha_u \text{ AND DCF } \delta_{vu} \text{ THEN } y_k = f_k\left(\mathcal{P}_V(v), \mathcal{P}_V(u)\right),$$

   where f_k is a trainable function.
3. **Normalization Layer:** Normalize the firing strengths r_k of each rule:

   $$\hat{r}_k = \frac{r_k}{\sum_{j=1}^{K} r_j}.$$

4. **Defuzzification Layer:** Aggregate the normalized rule outputs:

   $$y = \sum_{k=1}^{K} \hat{r}_k \cdot f_k(x).$$

Definition 2.18 (Multi-layer P-GNN)**.** For a multi-layer P-GNN, the l-th layer is defined as:

$$H^{(l)} = \sigma\left(f_\theta^{(l)}\left(H^{(l-1)}, A\right) + H^{(l-1)}\right),$$

where:

- $H^{(l)}$ is the feature matrix at layer l.
- A is the adjacency matrix of the graph.
- $f_\theta^{(l)}$ is a trainable function.
- σ is an activation function.

The final output is:

$$Y = \text{Softmax}\left(H^{(L)}\right),$$

where L is the number of layers.

Example 2.19 (Plithogenic Graph Neural Network (P-GNN): Marketing with Appurtenance and Contradiction)**.** Consider a marketing network where vertices represent consumers, and edges indicate potential "word-of-mouth" influence or co-purchasing patterns. Each consumer has multiple *plithogenic attributes*, such as:

- PriceSensitivity $\in [0,1]$
- BrandLoyalty $\in [0,1]$
- StylePreference $\in [0,1]$

We define $\mathcal{P}_V : \mathcal{X}_V \to [0,1]^s$ so that for a consumer v,

$$\mathcal{P}_V(x_v) = \ (\text{PriceSens}(v),\ \text{BrandLoyal}(v),\ \text{StylePref}(v),\ \dots),$$

where each component is the *degree of appurtenance (DAF)* in that attribute domain. Additionally, we may have a *contradiction function (DCF)* that measures how much two attribute values clash. For example, "high brand loyalty" might contradict "high price sensitivity."

Edges and Contradiction: For an edge (v,u), $\mathcal{P}_E(x_{v,u}) \in [0,1]^s$ might encode shared brand preferences or discount usage that link consumers v and u. A DCF, $\delta(\alpha_v, \alpha_u) \in [0,1]^t$, quantifies how contradictory α_v and α_u are along certain attributes.

Plithogenic Rules: A typical P-GNN rule might be:

$$\text{IF } v \text{ has } (\text{PriceSens}, \text{BrandLoyal}, \dots) \in \mathcal{P}_V(x_v) \quad \text{AND} \quad u \text{ has } (\text{PriceSens}, \text{BrandLoyal}, \dots) \in \mathcal{P}_V(x_u)$$

$$\text{AND } \delta_{vu} \approx 0 \quad \text{THEN} \quad \text{High synergy for cross-promotion.}$$

When δ_{vu} is small, there is little contradiction between the two consumers' preferences, so a marketing campaign might effectively target both simultaneously.

Normalization and Defuzzification: As in a fuzzy rule system, we compute rule firing strengths r_k, normalize them,

$$\hat{r}_k = \frac{r_k}{\sum_{j=1}^{K} r_j},$$

and output a crisp "marketing score." In a multi-layer P-GNN, each layer updates consumer representations ($H^{(l)}$) by combining neighbor data, but uses plithogenic logic (DAF + DCF) to weigh how contradictory neighbors' attributes are. The final output might be a Softmax($H^{(L)}$) classification for "likely to buy X" or "likely to join discount program," factoring in not only membership degrees but also the potential contradictions among consumer attributes.

Advantages: By merging standard GNN message-passing with plithogenic logic, P-GNN captures cases where even high membership in certain attributes can backfire if there is strong contradiction (e.g., "brand-loyal but extremely price-sensitive" might be contradictory for certain promotions). This extra dimension of DCF allows for more nuanced modeling of real-world consumer behavior than a purely fuzzy or crisp approach.

2.6 HyperUncertain Set

This subsection introduces the HyperUncertain Set and SuperHyperUncertain Set, which extend the concepts of Fuzzy Set, Neutrosophic Set, and Plithogenic Set using the frameworks of Hyperstructure and SuperHyperstructure.

2.6.1 Hyperstructure and Superhyperstructure

This subsubsection investigates the concepts of *Hyperstructure* and *Superhyperstructure*, which provide advanced mathematical frameworks for modeling hierarchical systems. A *Hyperstructure* extends the notion of the powerset, offering a formal mechanism to represent and analyze relationships among subsets of a given set [67,68]. Building on this idea, a *Superhyperstructure* employs the n-th powerset, thereby enabling the representation of multi-layered and hierarchical systems [45, 46, 69]. The formal definition of the n-th powerset is presented below.

Definition 2.20 (n-th Powerset)**.** (cf. [46, 70])

The n-th powerset of a set H, denoted by $\mathcal{P}_n(H)$, is defined recursively. Beginning with the standard powerset, the construction is given by:

$$\mathcal{P}_1(H) = \mathcal{P}(H), \qquad \mathcal{P}_{n+1}(H) = \mathcal{P}\big(\mathcal{P}_n(H)\big), \qquad n \geqslant 1.$$

Similarly, the n-th non-empty powerset, denoted by $\mathcal{P}_n^*(H)$, is defined as:

$$\mathcal{P}_1^*(H) = \mathcal{P}^*(H), \qquad \mathcal{P}_{n+1}^*(H) = \mathcal{P}^*\big(\mathcal{P}_n^*(H)\big),$$

where $\mathcal{P}^*(H)$ denotes the powerset of H with the empty set removed.

Example 2.21 (n-th Powerset of a Set)**.** Let $H = \{x, y\}$. The iterative construction of the n-th powerset is as follows:

- The first powerset ($\mathcal{P}_1(H)$) is the ordinary powerset:

$$\mathcal{P}_1(H) = \mathcal{P}(H) = \{\varnothing, \{x\}, \{y\}, \{x, y\}\}.$$

- The second powerset ($\mathcal{P}_2(H)$) is the powerset of $\mathcal{P}_1(H)$:

$$\mathcal{P}_2(H) = \mathcal{P}\big(\mathcal{P}_1(H)\big).$$

  Explicitly, its elements are subsets of $\mathcal{P}_1(H)$, for example

$$\{\varnothing\}, \quad \{\{x\}\}, \quad \{\{y\}\}, \quad \{\{x, y\}\}, \quad \{\varnothing, \{x\}\}, \ \ldots, \ \mathcal{P}_1(H).$$

Thus, each level $\mathcal{P}_n(H)$ introduces higher-order structures, with elements that are subsets of the previous powerset.

To provide a formal basis for the concepts of Hyperstructures and Superhyperstructures, we proceed with the following definitions.

Definition 2.22 (Hyperstructure). (cf. [46, 70]) A *Hyperstructure* extends the concept of a Classical Structure by operating on the powerset of a base set. Formally, it is defined as:

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ is its powerset, and $\circ$ represents an operation defined on subsets within $\mathcal{P}(S)$.

Definition 2.23 (n-Superhyperstructure). (cf. [46, 70]) An *n-Superhyperstructure* generalizes a Hyperstructure by utilizing the n-th powerset of a base set. It is formally represented as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ is the n-th powerset of S, and $\circ$ is an operation defined on elements of $\mathcal{P}_n(S)$.

2.6.2 Hyperfuzzy Set, and Superhyperfuzzy Set

This subsubsection introduces the definitions of Hyperfuzzy Set [42], and Superhyperfuzzy Set [44]. Intuitively, these concepts extend the idea of fuzzy values into hierarchical structures, enabling a more nuanced representation of uncertainty. The formal definitions are provided below.

Definition 2.24 (HyperFuzzy Set). [41, 71, 72] Let X be a non-empty set. A *HyperFuzzy Set* on X is a mapping

$$\tilde{\mu} : X \longrightarrow \tilde{\mathcal{P}}([0,1]),$$

where $\tilde{\mathcal{P}}([0,1])$ denotes the collection of all non-empty subsets of the unit interval $[0,1]$. For each element $x \in X$, the image $\tilde{\mu}(x) \subseteq [0,1]$ represents a set of possible membership degrees, rather than a single value.

Definition 2.25 (n-SuperHyperFuzzy Set). [43] Let X be a non-empty set. The n-SuperHyperFuzzy Set is a recursive generalization of fuzzy sets, hyperfuzzy sets, and superhyperfuzzy sets. It is defined as:

$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \geqslant 2$,
$$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$
represents the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1])$ is similarly defined for the interval $[0,1]$.
- $\tilde{\mu}_n$ assigns to each element $A \in \tilde{\mathcal{P}}_n(X)$ a non-empty subset $\tilde{\mu}_n(A) \subseteq [0,1]$, representing the degrees of membership associated with A at the n-th level.

Example 2.26 (Real-life example of an n-SuperHyperFuzzy Set (case $n = 2$)). Let

$$X = \{\text{Yamada},\ \text{Suzuki},\ \text{Tanaka}\}$$

be software engineers. We consider the fuzzy concept "readiness for 24/7 on-call duty." Elements of $\tilde{\mathcal{P}}_2(X)$ are nonempty sets of nonempty teams (i.e., bundles of candidate teams). For instance,

$$A := \big\{\{\text{Yamada}\},\ \{\text{Suzuki}, \text{Tanaka}\}\big\}, \qquad B := \big\{\{\text{Yamada}, \text{Suzuki}\},\ \{\text{Tanaka}\}\big\}.$$

Define

$$\tilde{\mu}_2 :\ \tilde{\mathcal{P}}_2(X) \ \longrightarrow\ \tilde{\mathcal{P}}_2([0,1])$$

by

$$\tilde{\mu}_2(A) = \big\{\{0.85\},\ \{0.55,\ 0.65\}\big\}, \qquad \tilde{\mu}_2(B) = \big\{\{0.70,\ 0.80\},\ \{0.40\}\big\}.$$

Here, each inner set (e.g., $\{0.55, 0.65\}$) collects membership degrees coming from distinct assessment scenarios (e.g., weekday vs. weekend load), while the outer braces aggregate alternative team bundles. Thus $\tilde{\mu}_2$ maps bundles of teams to bundles of membership degrees, as required by $\tilde{\mathcal{P}}_2([0,1])$, providing a concise multi-level representation of readiness under multiple policies.

2.6.3 HyperNeutrosophic and SuperHyperNeutrosophic Sets

Neutrosophic Sets extend Fuzzy Sets by incorporating the notion of indeterminacy, thereby allowing the representation of situations that are neither entirely true nor entirely false [29]. This generalization provides a more comprehensive framework for modeling real-world scenarios characterized by heightened uncertainty and complexity, surpassing the capabilities of traditional Fuzzy Sets; accordingly, Neutrosophic Sets have attracted substantial attention in diverse studies [30, 73, 74]. This idea has been further developed via HyperStructure and SuperHyperStructure, leading to the definitions of HyperNeutrosophic Sets and SuperHyperNeutrosophic Sets, presented below [75].

Definition 2.27 (HyperNeutrosophic Set)**.** [43, 44] Let X be a nonempty set. A mapping

$$\tilde{\mu} : X \longrightarrow \tilde{\mathcal{P}}\big([0,1]^3\big)$$

is called a *HyperNeutrosophic Set* on X, where $\tilde{\mathcal{P}}\big([0,1]^3\big)$ denotes the family of all nonempty subsets of the unit cube $[0,1]^3$. For each $x \in X$, the image $\tilde{\mu}(x) \subseteq [0,1]^3$ is a set of neutrosophic membership triples (T, I, F) (truth, indeterminacy, falsity) satisfying

$$0 \leqslant T + I + F \leqslant 3.$$

Definition 2.28 (n-SuperHyperNeutrosophic Set)**.** [43, 44] Let X be a nonempty set. An *n-SuperHyperNeutrosophic Set* is a recursive generalization of Neutrosophic Sets, HyperNeutrosophic Sets, and SuperHyperNeutrosophic Sets, defined by a mapping

$$\tilde{A}_n : \tilde{\mathcal{P}}_n(X) \longrightarrow \tilde{\mathcal{P}}_n\big([0,1]^3\big),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \geqslant 2$,
$$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}\big(\tilde{\mathcal{P}}_{k-1}(X)\big)$$
is the k-th nested family of nonempty subsets of X;
- $\tilde{\mathcal{P}}_n\big([0,1]^3\big)$ is defined analogously over $[0,1]^3$;
- for each $A \in \tilde{\mathcal{P}}_n(X)$, the value $\tilde{A}_n(A) \subseteq [0,1]^3$ collects the neutrosophic degrees (T, I, F) associated with the n-th level subsets of X.

For every $A \in \tilde{\mathcal{P}}_n(X)$ and every $(T, I, F) \in \tilde{A}_n(A)$, the constraint

$$0 \leqslant T + I + F \leqslant 3$$

holds, where T, I, and F denote the truth, indeterminacy, and falsity components, respectively.

Example 2.29 (Real-life example of an n-SuperHyperNeutrosophic Set ($n = 2$))**.** Let

$$X = \{\text{Yamada},\ \text{Suzuki},\ \text{Tanaka}\}$$

be field teams for network recovery. We assess the concept "ability to restore service within 2 hours" under multiple disaster scenarios and expert panels. Elements of $\tilde{\mathcal{P}}_2(X)$ are nonempty sets of nonempty teams (bundles of candidate teams). For instance,

$$A = \big\{\{\text{Yamada}\},\ \{\text{Suzuki}, \text{Tanaka}\}\big\}, \qquad B = \big\{\{\text{Yamada}, \text{Tanaka}\}\big\}.$$

Define the 2-SuperHyperNeutrosophic mapping

$$\tilde{A}_2 :\ \tilde{\mathcal{P}}_2(X) \ \longrightarrow\ \tilde{\mathcal{P}}_2\big([0,1]^3\big)$$

by assigning sets of neutrosophic triples (T, I, F) (truth, indeterminacy, falsity) grouped by scenario/panel:

$$\tilde{A}_2(A) = \Big\{ \{(0.82,\, 0.10,\, 0.08)\},\ \{(0.68,\, 0.22,\, 0.10),\ (0.60,\, 0.25,\, 0.15)\} \Big\},$$
$$\tilde{A}_2(B) = \Big\{ \{(0.75,\, 0.15,\, 0.10)\},\ \{(0.70,\, 0.20,\, 0.10)\} \Big\}.$$

Interpretation: the outer braces aggregate alternative "bundles of assessments" (e.g., different disaster types or policies), while each inner set collects expert-panel evaluations for that bundle. All entries satisfy $0 \leqslant T + I + F \leqslant 3$ (e.g., $0.82 + 0.10 + 0.08 = 1.00$, $0.68 + 0.22 + 0.10 = 1.00$), as required. Thus $\tilde{A}_2$ maps bundles of teams to bundles of neutrosophic degrees, capturing multi-level uncertainty across scenarios and expert groups.

2.6.4 HyperPlithogenic Set

The HyperPlithogenic Set is a generalized concept of the Plithogenic Set. Furthermore, an even broader generalization, known as the SuperHyperPlithogenic Set, has been established. The definitions and related concepts are detailed below.

Definition 2.30 (HyperPlithogenic Set)**.** [43] Let X be a non-empty set, and let A be a set of attributes. For each attribute $v \in A$, let Pv be the set of possible values of v. A *HyperPlithogenic Set* HPS over X is defined as:

$$HPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF)$$

where:

- $P \subseteq X$ is a subset of the universe.
- For each attribute v_i, Pv_i is the set of possible values.
- For each attribute v_i, $\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P$ and attribute value $a_i \in Pv_i$ a set of membership degrees.
- $pCF : (\bigcup_{i=1}^n Pv_i) \times (\bigcup_{i=1}^n Pv_i) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

Definition 2.31 (n-SuperHyperPlithogenic Set)**.** [43] Let X be a non-empty set, and let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of attributes, each associated with a set of possible values P_{v_i}. An n-SuperHyperPlithogenic Set ($SHPS_n$) is defined recursively as:

$$SHPS_n = (P_n, V, \{P_{v_i}\}_{i=1}^n, \{\tilde{pdf}_i^{(n)}\}_{i=1}^n, pCF^{(n)}),$$

where:

- $P_1 \subseteq X$, and for $k \geqslant 2$,
$$P_k = \tilde{\mathcal{P}}(P_{k-1}),$$
represents the k-th nested family of non-empty subsets of P_1.
- For each attribute $v_i \in V$, P_{v_i} is the set of possible values of the attribute v_i.
- For each k-th level subset P_k, $\tilde{pdf}_i^{(n)} : P_n \times P_{v_i} \to \tilde{\mathcal{P}}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P_n$ and attribute value $a_i \in P_{v_i}$ a subset of $[0,1]^s$.
- $pCF^{(n)} : \bigcup_{i=1}^n P_{v_i} \times \bigcup_{i=1}^n P_{v_i} \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*, satisfying:

1. Reflexivity: $pCF^{(n)}(a,a) = 0$ for all $a \in \bigcup_{i=1}^{n} P_{v_i}$,
2. Symmetry: $pCF^{(n)}(a,b) = pCF^{(n)}(b,a)$ for all $a, b \in \bigcup_{i=1}^{n} P_{v_i}$.

- s and t are positive integers representing the dimensions of the membership degrees and contradiction degrees, respectively.

Example 2.32 (Real-life example of an n-SuperHyperPlithogenic Set ($n = 2$)). Let

$$X = \{\text{Yamada, Suzuki, Tanaka}\}$$

be engineers for a project. Choose two attributes

$$V = \{v_1 = \text{Experience}, \ v_2 = \text{Availability}\}, \quad P_{v_1} = \{\text{Junior, Senior}\}, \quad P_{v_2} = \{\text{Weekday, Weekend}\}.$$

Set $P_1 := X$ and $P_2 := \tilde{\mathcal{P}}(P_1)$ (all nonempty teams). Consider

$$A = \{\text{Yamada}, \text{Suzuki}\}, \qquad B = \{\text{Tanaka}\} \ \in P_2.$$

Take $s = t = 1$. Define the HDAFs $\tilde{pdf}_i^{(2)} : P_2 \times P_{v_i} \to \tilde{\mathcal{P}}([0,1])$ by

$$\begin{aligned}
\tilde{pdf}_1^{(2)}(A, \text{Senior}) &= \{0.82, 0.88\}, & \tilde{pdf}_1^{(2)}(A, \text{Junior}) &= \{0.35, 0.50\},\\
\tilde{pdf}_2^{(2)}(A, \text{Weekday}) &= \{0.90\}, & \tilde{pdf}_2^{(2)}(A, \text{Weekend}) &= \{0.60, 0.70\},\\
\tilde{pdf}_1^{(2)}(B, \text{Senior}) &= \{0.60\}, & \tilde{pdf}_1^{(2)}(B, \text{Junior}) &= \{0.75\},\\
\tilde{pdf}_2^{(2)}(B, \text{Weekday}) &= \{0.55\}, & \tilde{pdf}_2^{(2)}(B, \text{Weekend}) &= \{0.80\}.
\end{aligned}$$

Define the DCF $pCF^{(2)} : \left(P_{v_1} \cup P_{v_2}\right)^2 \to [0,1]$ (symmetric, reflexive) by

$$pCF^{(2)}(\text{Junior}, \text{Senior}) = 0.7, \quad pCF^{(2)}(\text{Weekday}, \text{Weekend}) = 0.6, \quad pCF^{(2)}(\alpha, \beta) = 0 \text{ otherwise},$$

and $pCF^{(2)}(a,a) = 0$ for all a. Then

$$SHPS_2 = \left(P_2, \ V, \ \{P_{v_i}\}_{i=1}^{2}, \ \{\tilde{pdf}_i^{(2)}\}_{i=1}^{2}, \ pCF^{(2)}\right)$$

models team suitability under multi-attribute, contradiction-aware evaluations, where each team (from P_2) receives *sets* of membership degrees and inter-value contradictions are explicitly quantified.

3 Result: HyperUncertain Graph and SuperHyperUncertain Graph

As a result of this paper, we investigate the concepts of HyperUncertain Graph and SuperHyperUncertain Graph.

3.1 HyperFuzzy Graph

The HyperFuzzy Graph is a graph-based extension of the HyperFuzzy Set, while the SuperHyperFuzzy Graph is a graph-based extension of the SuperHyperFuzzy Set. Definitions and related concepts are provided below.

Definition 3.1 (Hyperfuzzy Graph). [43] A *hyperfuzzy graph* is a quadruple

$$G_H = \left(V, \ E, \ \tilde{\sigma}, \ \tilde{\mu}\right),$$

where

1. V is a non-empty finite set of vertices,
2. $E \subseteq V \times V$ is a set of edges,
3. $\tilde{\sigma} : V \longrightarrow \tilde{\mathcal{P}}_1\big([0,1]\big)$ assigns to each vertex $v \in V$ a non-empty subset $\tilde{\sigma}(v) \subseteq [0,1]$,

4. $\tilde{\mu} : E \longrightarrow \tilde{\mathcal{P}}_1\big([0,1]\big)$ assigns to each edge $e = (u,v) \in E$ a non-empty subset $\tilde{\mu}(e) \subseteq [0,1]$.

Here, $\tilde{\mathcal{P}}_1([0,1]) = \{A \subseteq [0,1] \mid A \neq \varnothing\}$. Moreover, the following *consistency condition* must hold:

$$\sup\big(\tilde{\mu}(u,v)\big) \leqslant \min\big\{\sup\big(\tilde{\sigma}(u)\big),\ \sup\big(\tilde{\sigma}(v)\big)\big\} \quad \text{for all } (u,v) \in E,$$

where for $A \subseteq [0,1]$, $\sup(A) = \max A$.

The *underlying crisp graph* $G_H^* = (V^*, E^*)$ is given by

$$V^* = \{v \in V \mid \sup(\tilde{\sigma}(v)) > 0\}, \quad E^* = \{e \in E \mid \sup(\tilde{\mu}(e)) > 0\}.$$

Definition 3.2 (n-SuperHyperfuzzy Graph)**.** [43] Let $n \geqslant 1$. Define recursively

$$\tilde{\mathcal{P}}_1([0,1]) = \{A \subseteq [0,1] : A \neq \varnothing\}, \quad \tilde{\mathcal{P}}_k([0,1]) = \{B \subseteq \tilde{\mathcal{P}}_{k-1}([0,1]) : B \neq \varnothing\},\ k \geqslant 2.$$

An *n-SuperHyperfuzzy graph* is a quadruple

$$G_{nH} = \big(V,\, E,\, \tilde{\sigma}_n,\, \tilde{\mu}_n\big),$$

where

1. V is a non-empty finite vertex set,
2. $E \subseteq V \times V$ is the edge set,
3. $\tilde{\sigma}_n : V \to \tilde{\mathcal{P}}_n([0,1])$ assigns to each $v \in V$ a non-empty nested set of depth n,
4. $\tilde{\mu}_n : E \to \tilde{\mathcal{P}}_n([0,1])$ assigns to each $e \in E$ a non-empty nested set of depth n.

Introduce the operator $\sup_k : \tilde{\mathcal{P}}_k([0,1]) \to [0,1]$ by

$$\sup_1(A) = \max A, \quad \sup_k(B) = \max\big\{\sup_{k-1}(A) \mid A \in B\big\},\ k \geqslant 2.$$

Then the *consistency condition* reads:

$$\sup_n\big(\tilde{\mu}_n(u,v)\big) \leqslant \min\big\{\sup_n\big(\tilde{\sigma}_n(u)\big),\ \sup_n\big(\tilde{\sigma}_n(v)\big)\big\} \quad \forall\, (u,v) \in E.$$

The *underlying crisp graph* $G_{nH}^* = (V^*, E^*)$ is defined by

$$V^* = \{v \in V : \sup_n(\tilde{\sigma}_n(v)) > 0\}, \quad E^* = \{e \in E : \sup_n(\tilde{\mu}_n(e)) > 0\}.$$

Example 3.3 (2-SuperHyperfuzzy Graph)**.** Let $n = 2$,

$$V = \{v_1, v_2\}, \qquad E = \{(v_1, v_2)\}.$$

Recall $\tilde{\mathcal{P}}_1([0,1])$ are nonempty subsets of $[0,1]$, and $\tilde{\mathcal{P}}_2([0,1])$ are nonempty families thereof. Define

$$\tilde{\sigma}_2(v_1) = \big\{\{0.6\}, \{0.5, 0.8\}\big\}, \quad \tilde{\sigma}_2(v_2) = \big\{\{0.7\}\big\},$$

and

$$\tilde{\mu}_2(v_1, v_2) = \big\{\{0.5\}, \{0.6\}\big\}.$$

Here

$$\sup_2(\tilde{\sigma}_2(v_1)) = \max\{\max\{0.6\}, \max\{0.5, 0.8\}\} = 0.8, \quad \sup_2(\tilde{\sigma}_2(v_2)) = 0.7,$$

$$\sup_2(\tilde{\mu}_2(v_1, v_2)) = \max\{0.5, 0.6\} = 0.6,$$

and indeed

$$0.6 = \sup_2(\tilde{\mu}_2(v_1, v_2)) \leqslant \min\{0.8,\ 0.7\} = 0.7.$$

Thus $(V, E, \tilde{\sigma}_2, \tilde{\mu}_2)$ is a valid 2-SuperHyperfuzzy graph.

Theorem 3.4. *Let $G_{nH} = (V, E, \tilde{\sigma}_n, \tilde{\mu}_n)$ be the n-SuperHyperfuzzy graph of Definition 3.2. Then:*

1. *If $n = 1$, G_{1H} coincides exactly with the Hyperfuzzy graph $G_H = (V, E, \tilde{\sigma}, \tilde{\mu})$ of Definition 3.1.*
2. *Conversely, any Hyperfuzzy graph $G_H = (V, E, \tilde{\sigma}, \tilde{\mu})$ can be regarded as the $n = 1$ case of an n-SuperHyperfuzzy graph by setting $\tilde{\sigma}_1(v) = \tilde{\sigma}(v)$ and $\tilde{\mu}_1(e) = \tilde{\mu}(e)$.*

Proof. **(1) Reduction when $n = 1$.** By definition,

$$\tilde{\mathcal{P}}_1([0,1]) = \big\{A \subseteq [0,1] \mid A \neq \varnothing\big\}.$$

Hence a function $\tilde{\sigma}_1 : V \to \tilde{\mathcal{P}}_1([0,1])$ is exactly the same data as $\tilde{\sigma} : V \to \tilde{\mathcal{P}}_1([0,1])$ in Definition 3.1. Similarly, $\tilde{\mu}_1 : E \to \tilde{\mathcal{P}}_1([0,1])$ matches $\tilde{\mu} : E \to \tilde{\mathcal{P}}_1([0,1])$. Moreover, the consistency condition

$$\sup_1\big(\tilde{\mu}_1(u,v)\big) = \max \tilde{\mu}(u,v) \;\leqslant\; \min\big\{\max \tilde{\sigma}(u),\, \max \tilde{\sigma}(v)\big\}$$

is precisely the requirement $\sup(\tilde{\mu}(u,v)) \leqslant \min\{\sup(\tilde{\sigma}(u)), \sup(\tilde{\sigma}(v))\}$ from Definition 3.1. Thus all data and conditions agree, and $G_{1H} = G_H$.

(2) Embedding a Hyperfuzzy graph as $n = 1$. Conversely, let

$$G_H = (V, E, \tilde{\sigma}, \tilde{\mu})$$

be any Hyperfuzzy graph. Define

$$\tilde{\sigma}_1(v) := \tilde{\sigma}(v), \quad \tilde{\mu}_1(e) := \tilde{\mu}(e),$$

viewing each $\tilde{\sigma}(v)$ and $\tilde{\mu}(e)$ as a nonempty subset of $[0,1]$. Then $(V, E, \tilde{\sigma}_1, \tilde{\mu}_1)$ satisfies exactly the requirements of Definition 3.2 with $n = 1$. In particular, the consistency condition

$$\sup_1\big(\tilde{\mu}_1(u,v)\big) = \max \tilde{\mu}(u,v) \;\leqslant\; \min\{\max \tilde{\sigma}(u),\, \max \tilde{\sigma}(v)\}$$

holds by hypothesis. Therefore G_H is exactly the 1-SuperHyperfuzzy graph G_{1H}. □

3.2 Hyperneutrosophic Graph

The Hyperneutrosophic Graph is a graph-based extension of the Hyperneutrosophic Set, while the SuperHyperneutrosophic Graph is a graph-based extension of the SuperHyperneutrosophic Set. Definitions and related concepts are provided below.

Definition 3.5 (Hyperneutrosophic Graph)**.** Let V be a nonempty finite vertex set and $E \subseteq V \times V$ an edge set. Denote by

$$\tilde{\mathcal{P}}_1\big([0,1]^3\big) = \big\{A \subseteq [0,1]^3 : A \neq \varnothing\big\}.$$

A *Hyperneutrosophic Graph* is a quadruple

$$G_{HN} = (\,V,\; E,\; \tilde{\sigma},\; \tilde{\mu}\,),$$

where

- $\tilde{\sigma} : V \to \tilde{\mathcal{P}}_1([0,1]^3)$ assigns to each vertex v a nonempty set of neutrosophic triplets $\big\{(T, I, F) \in [0,1]^3\big\}$ satisfying $0 \leqslant T + I + F \leqslant 3$.
- $\tilde{\mu} : E \to \tilde{\mathcal{P}}_1([0,1]^3)$ assigns to each edge e a nonempty set of triplets (T, I, F) with $0 \leqslant T+I+F \leqslant 3$.

Define the coordinate-wise suprema for $A \subseteq [0,1]^3$:

$$\overset{T}{\sup}(A) = \sup_{(T,I,F)\in A} T, \quad \overset{I}{\sup}(A) = \sup_{(T,I,F)\in A} I, \quad \overset{F}{\sup}(A) = \sup_{(T,I,F)\in A} F.$$

We require the following *consistency* for every edge $e = (u,v) \in E$:

$$\overset{j}{\sup}\big(\tilde{\mu}(e)\big) \leqslant \min\Big\{\overset{j}{\sup}\big(\tilde{\sigma}(u)\big),\ \overset{j}{\sup}\big(\tilde{\sigma}(v)\big)\Big\} \quad \text{for } j \in \{T,I,F\}.$$

The *underlying crisp graph* is

$$V^* = \{v : \overset{T}{\sup}(\tilde{\sigma}(v)) > 0\}, \quad E^* = \{e : \overset{T}{\sup}(\tilde{\mu}(e)) > 0\}.$$

Example 3.6 (Hyperneutrosophic Graph)**.** Let

$$V = \{v_1, v_2\}, \qquad E = \{e = (v_1, v_2)\}.$$

We work in $\tilde{\mathcal{P}}_1([0,1]^3)$, the family of all nonempty subsets of $[0,1]^3$. Define

$$\tilde{\sigma}(v_1) = \big\{(0.7,\ 0.2,\ 0.1),\ (0.5,\ 0.4,\ 0.1)\big\}, \qquad \tilde{\sigma}(v_2) = \big\{(0.6,\ 0.3,\ 0.2)\big\},$$

and

$$\tilde{\mu}(e) = \big\{(0.5,\ 0.2,\ 0.1)\big\}.$$

For any $A \subseteq [0,1]^3$, define coordinate-wise suprema

$$\overset{T}{\sup}(A) = \sup_{(T,I,F)\in A} T, \quad \overset{I}{\sup}(A) = \sup_{(T,I,F)\in A} I, \quad \overset{F}{\sup}(A) = \sup_{(T,I,F)\in A} F.$$

Then

$$\overset{T}{\sup}\big(\tilde{\sigma}(v_1)\big) = 0.7,\ \overset{I}{\sup}\big(\tilde{\sigma}(v_1)\big) = 0.4,\ \overset{F}{\sup}\big(\tilde{\sigma}(v_1)\big) = 0.1,$$
$$\overset{T}{\sup}\big(\tilde{\sigma}(v_2)\big) = 0.6,\ \overset{I}{\sup}\big(\tilde{\sigma}(v_2)\big) = 0.3,\ \overset{F}{\sup}\big(\tilde{\sigma}(v_2)\big) = 0.2,$$
$$\overset{T}{\sup}\big(\tilde{\mu}(e)\big) = 0.5,\ \overset{I}{\sup}\big(\tilde{\mu}(e)\big) = 0.2,\ \overset{F}{\sup}\big(\tilde{\mu}(e)\big) = 0.1.$$

One checks the consistency condition for each coordinate $j \in \{T,I,F\}$:

$$\overset{j}{\sup}\big(\tilde{\mu}(e)\big) \ = \ (0.5,\ 0.2,\ 0.1) \ \leqslant \ \min\{(0.7,\ 0.4,\ 0.1),\ (0.6,\ 0.3,\ 0.2)\},$$

so $(V, E, \tilde{\sigma}, \tilde{\mu})$ is a valid Hyperneutrosophic Graph.

Definition 3.7 (n-SuperHyperneutrosophic Graph)**.** Fix $n \geqslant 1$. Recursively define $\tilde{\mathcal{P}}_k([0,1]^3)$ as the family of all nonempty subsets of $\tilde{\mathcal{P}}_{k-1}([0,1]^3)$. An *$n$-SuperHyperneutrosophic Graph* is

$$G_{nHN} = \big(V,\ E,\ \tilde{\sigma}_n,\ \tilde{\mu}_n\big),$$

with

$$\tilde{\sigma}_n : V \to \tilde{\mathcal{P}}_n([0,1]^3), \quad \tilde{\mu}_n : E \to \tilde{\mathcal{P}}_n([0,1]^3)],$$

subject to: for each $(u,v) \in E$,

$$\overset{j}{\underset{n}{\sup}}\big(\tilde{\mu}_n(u,v)\big) \ \leqslant \ \min\Big\{\overset{j}{\underset{n}{\sup}}\big(\tilde{\sigma}_n(u)\big),\ \overset{j}{\underset{n}{\sup}}\big(\tilde{\sigma}_n(v)\big)\Big\}, \quad j \in \{T,I,F\},$$

where $\sup_n^j$ is the recursive supremum on the j-th coordinate.

Example 3.8 (2–SuperHyperneutrosophic Graph)**.** Let again

$$V = \{v_1, v_2\}, \quad E = \{e = (v_1, v_2)\},$$

and work in $\tilde{\mathcal{P}}_2([0,1]^3)$, the family of nonempty subsets of $\tilde{\mathcal{P}}_1([0,1]^3)$. Define

$$\tilde{\sigma}_2(v_1) = \Big\{\underbrace{\{\underbrace{(0.7,0.2,0.1)}, \underbrace{(0.5,0.4,0.1)}\}}_{\in\tilde{\mathcal{P}}_1},\ \underbrace{\{\underbrace{(0.6,0.3,0.2)}\}}_{\in\tilde{\mathcal{P}}_1}\Big\},$$

$$\tilde{\sigma}_2(v_2) = \big\{ (0.6, 0.3, 0.1) \big\}, \quad \tilde{\mu}_2(e) = \big\{ \{(0.5, 0.2, 0.1)\},\ \{(0.4, 0.3, 0.2), (0.5, 0.4, 0.1)\} \big\}.$$

We define recursively for $A \in \tilde{\mathcal{P}}_2$:

$$\sup_2^j(A) = \sup\big\{ \sup_1^j(B) \mid B \in A \big\}, \quad \sup_1^j(B) = \sup_{(T,I,F)\in B} j, \quad j \in \{T, I, F\}.$$

Then one computes:

$$\sup_2^T\big(\tilde{\sigma}_2(v_1)\big) = \max\{\, \max\{0.7, 0.5\},\ 0.6\} = 0.7, \quad \sup_2^T\big(\tilde{\sigma}_2(v_2)\big) = 0.6,$$

$$\sup_2^T\big(\tilde{\mu}_2(e)\big) = \max\{\, 0.5,\ \max\{0.4, 0.5\}\} = 0.5,$$

and similarly for the I- and F-coordinates. Since

$$0.5 \ \leqslant\ \min\{0.7,\ 0.6\},$$

and analogously for the other two coordinates, the consistency condition holds. Hence $(V, E, \tilde{\sigma}_2, \tilde{\mu}_2)$ is a valid 2–SuperHyperneutrosophic Graph.

Theorem 3.9. *Let $n \geqslant 1$, and let*

$$G_{nHN} = \big(V,\ E,\ \tilde{\sigma}_n,\ \tilde{\mu}_n\big)$$

be an n-SuperHyperneutrosophic Graph as in Definition 3.7. Then:

1. *If $n = 1$, the structure $(V, E, \tilde{\sigma}_1, \tilde{\mu}_1)$ satisfies exactly the axioms of a Hyperneutrosophic Graph (Definition 3.5).*
2. *Conversely, any Hyperneutrosophic Graph $(V, E, \tilde{\sigma}, \tilde{\mu})$ can be viewed as the $n = 1$ case of an n-SuperHyperneutrosophic Graph by setting $\tilde{\sigma}_1(v) = \tilde{\sigma}(v)$ and $\tilde{\mu}_1(e) = \tilde{\mu}(e)$.*

Proof. Recall that by definition

$$\tilde{\mathcal{P}}_1\big([0,1]^3\big) = \big\{A \subseteq [0,1]^3 : A \neq \varnothing\big\},$$

so for $n = 1$ the "super-" nesting disappears. Concretely:

(1) The maps $\tilde{\sigma}_1 : V \to \tilde{\mathcal{P}}_1([0,1]^3)$ and $\tilde{\mu}_1 : E \to \tilde{\mathcal{P}}_1([0,1]^3)$ are then precisely the vertex and edge assignments required in Definition 3.5. Moreover, the recursive supremum $\sup_n^j$ on the j–th coordinate reduces to $\sup_1^j(A) = \max\{a_j : (a_T, a_I, a_F) \in A\}$, which is exactly the coordinatewise sup used in the Hyperneutrosophic consistency condition

$$\sup_1^j\big(\tilde{\mu}_1(u, v)\big) \leqslant \min\big\{ \sup_1^j(\tilde{\sigma}_1(u)),\ \sup_1^j(\tilde{\sigma}_1(v)) \big\}.$$

Hence all data and constraints agree, so G_{1HN} is a Hyperneutrosophic Graph.

(2) Conversely, let $(V, E, \tilde{\sigma}, \tilde{\mu})$ be any Hyperneutrosophic Graph. Define

$$\tilde{\sigma}_1(v) := \tilde{\sigma}(v), \qquad \tilde{\mu}_1(e) := \tilde{\mu}(e).$$

Since $\tilde{\sigma}(v)$ and $\tilde{\mu}(e)$ already lie in $\tilde{\mathcal{P}}_1([0,1]^3)$ and satisfy the required consistency inequalities, the quadruple $\big(V, E, \tilde{\sigma}_1, \tilde{\mu}_1\big)$ meets all conditions of an n-SuperHyperneutrosophic Graph with $n = 1$. □

Theorem 3.10. *Every Hyperfuzzy Graph $G_H = (V, E, \tilde{\sigma}^f, \tilde{\mu}^f)$ (Definition 3.1) can be embedded into a Hyperneutrosophic Graph by representing each fuzzy membership $t \in [0,1]$ as the neutrosophic triple $(t, 0, 1-t)$. Conversely, any Hyperneutrosophic Graph whose images lie in $\{(t, 0, 1-t) \mid t \in [0,1]\}$ is a Hyperfuzzy Graph.*

Proof. Given G_H, define

$$\tilde{\sigma}(v) = \big\{(t, 0, 1-t) \mid t \in \tilde{\sigma}^f(v)\big\}, \quad \tilde{\mu}(e) = \big\{(t, 0, 1-t) \mid t \in \tilde{\mu}^f(e)\big\}.$$

Then $\sup^T(\tilde{\sigma}(v)) = \sup \tilde{\sigma}^f(v)$ and $\sup^F(\tilde{\sigma}(v)) = \sup\{1-t \mid t \in \tilde{\sigma}^f(v)\} = 1 - \inf \tilde{\sigma}^f(v)$, while $\sup^I = 0$. The consistency $\sup^T(\tilde{\mu}(e)) \leqslant \min\{\sup^T(\tilde{\sigma}(u)), \sup^T(\tilde{\sigma}(v))\}$ is exactly the hyperfuzzy condition. Hence G_{HN} is a Hyperneutrosophic Graph.

Conversely, if in a Hyperneutrosophic Graph every image set consists only of triples $(t, 0, 1-t)$, projecting onto the first coordinate recovers a Hyperfuzzy Graph. □

Theorem 3.11. *Every SuperHyperfuzzy Graph (Definition 3.2) embeds into the n-SuperHyperneutrosophic Graph of Definition 3.7 via the same mapping $t \mapsto (t, 0, 1-t)$. Conversely, any n-SuperHyperneutrosophic Graph whose vertex and edge images lie in $\{\{(t, 0, 1-t)\} \mid t \in [0, 1]\} \subset \tilde{\mathcal{P}}_n([0, 1]^3)$ is a SuperHyperfuzzy Graph.*

Proof. Let

$$G_{nN} \;=\; \big(V,\; E,\; \tilde{\sigma}_n,\; \tilde{\mu}_n\big)$$

be an n-SuperHyperNeutrosophic Graph as in the Definition. We will construct a corresponding n-SuperHyperPlithogenic Graph

$$PG_{nH} \;=\; (PM_{nH},\, PN_{nH})$$

and verify all its HyperPlithogenic constraints.

1. Coordinate projections at each level. Recall that $\tilde{\sigma}_n(v) \in \tilde{\mathcal{P}}_n([0, 1]^3)$ is a nonempty nested family of depth n whose "atoms" at the bottom level are triples $(T, I, F) \in [0, 1]^3$ with $0 \leqslant T+I+F \leqslant 3$. For each coordinate $j \in \{T, I, F\}$, define the projection

$$\pi_j : [0, 1]^3 \;\longrightarrow\; [0, 1], \qquad \pi_j(T, I, F) \;=\; \begin{cases} T, & j = T, \\ I, & j = I, \\ F, & j = F. \end{cases}$$

This extends recursively to

$$\pi_j^{(k)} : \tilde{\mathcal{P}}_k([0, 1]^3) \;\longrightarrow\; \tilde{\mathcal{P}}_k([0, 1])$$

by applying π_j to every bottom-level triple inside the k-nested structure.

2. Defining the HyperPlithogenic DAFs. Let $Ml = \{T, I, F\}$. For each vertex $v \in V$ and each $j \in Ml$, set

$$adf_{nH}(v, j) \;:=\; \pi_j^{(n)}\big(\tilde{\sigma}_n(v)\big) \;\in\; \tilde{\mathcal{P}}_n([0, 1]).$$

Similarly, for each edge $e \in E$ and $j \in Nm = \{T, I, F\}$, define

$$bdf_{nH}(e, j) \;:=\; \pi_j^{(n)}\big(\tilde{\mu}_n(e)\big) \;\in\; \tilde{\mathcal{P}}_n([0, 1]).$$

By construction, each of these is nonempty, since $\tilde{\sigma}_n(v)$ and $\tilde{\mu}_n(e)$ are nonempty and π_j is surjective onto its image.

3. Supremum operators coincide. Define the coordinate-wise "nested supremum" on $\tilde{\mathcal{P}}_k([0, 1])$:

$$\sup_k^{\mathrm{cw}}(A) = \begin{cases} \max A, & k = 1, \\ \max\big\{\sup_{\mathrm{cw}\,k-1}(B) \mid B \in A\big\}, & k \geqslant 2. \end{cases}$$

One checks by induction on k that

$$\sup_n^{\mathrm{cw}}\big(\pi_j^{(n)}(A)\big) \;=\; \max\big\{\pi_j(x) \mid x \in A_{\mathrm{bottom}}\big\} \;=\; \sup_n^{j}(A),$$

where $A_{\text{bottom}} \subseteq [0,1]^3$ is the set of all bottom-level triples in A, and $\sup_n^j$ is the usual nested-supremum on the j-th component in the neutrosophic graph.

4. Appurtenance constraint. The original neutrosophic graph satisfies, for every edge $e = (u, v)$ and each coordinate j,

$$\sup_n^j\big(\tilde{\mu}_n(e)\big) \;\leqslant\; \min\Big\{\sup_n^j\big(\tilde{\sigma}_n(u)\big),\; \sup_n^j\big(\tilde{\sigma}_n(v)\big)\Big\}.$$

By step 3, this is equivalent to

$$\sup_{n,\text{cw}}\big(bdf_{nH}(e,j)\big) \;\leqslant_{\text{cw}}\; \min\Big\{\sup_{n,\text{cw}}\big(adf_{nH}(u,j)\big),\; \sup_{n,\text{cw}}\big(adf_{nH}(v,j)\big)\Big\},$$

which is exactly the HyperPlithogenic appurtenance constraint of Definition 3.13 (applied coordinate-wise).

5. Contradiction functions. Define

$$aCf(j,j') = 0, \quad bCf(j,j') = 0, \qquad \forall\, j, j' \in \{T, I, F\}.$$

These satisfy reflexivity and symmetry. Since the neutrosophic graph imposes no further contradiction constraints, the plithogenic contradiction constraint

$$bCf\big((j_1,j_2),(k_1,k_2)\big) \;\leqslant_{\text{cw}}\; \min\{aCf(j_1,k_1),\, aCf(j_2,k_2)\}$$

holds trivially.

Thus all HyperPlithogenic Graph axioms are satisfied, and we have exhibited an embedding of G_{nN} into PG_{nH}. □

3.3 HyperPlithogenic Graph

The HyperPlithogenic Graph is a graph-based extension of the HyperPlithogenic Set, while the SuperHyperPlithogenic Graph is a graph-based extension of the SuperHyperPlithogenic Set. Definitions and related concepts are provided below.

Definition 3.12 (HyperPlithogenic Graph)**.** Let $G = (V, E)$ be a finite simple graph. Let

$$Ml\ ,\ Nm$$

be finite sets of *vertex* and *edge* attribute values, respectively. Denote by

$$\tilde{\mathcal{P}}_1\big([0,1]^s\big) = \big\{A \subseteq [0,1]^s : A \neq \varnothing\big\}.$$

A *HyperPlithogenic Graph* is a pair

$$PG_H = \big(PM_H,\, PN_H\big),$$

where

$$PM_H = (M,\, l,\, Ml,\, adf_H,\, aCf), \qquad PN_H = (N,\, m,\, Nm,\, bdf_H,\, bCf),$$

satisfy:

1. $M \subseteq V$ is the vertex subset; l is a vertex-attribute with value-set Ml.
   - $adf_H : M \times Ml \to \tilde{\mathcal{P}}_1([0,1]^s)$ assigns each (x, a) a nonempty set of s-vectors in $[0,1]^s$.
   - $aCf : Ml \times Ml \to [0,1]^t$ is symmetric and reflexive:

   $$aCf(a,a) = \mathbf{0}, \quad aCf(a,b) = aCf(b,a).$$

2. $N \subseteq E$ is the edge subset; m is an edge-attribute with values Nm.

- $bdf_H : N \times Nm \to \tilde{\mathcal{P}}_1([0,1]^s)$ assigns each (e,b) a nonempty set in $[0,1]^s$.
- $bCf : Nm \times Nm \to [0,1]^t$ is symmetric and reflexive:

$$bCf(b,b) = \mathbf{0}, \quad bCf(b_1,b_2) = bCf(b_2,b_1).$$

3. *Hyper-appurtenance constraint.* Define the coordinate-wise supremum on $A \subseteq [0,1]^s$ by

$$\sup_{1,\mathrm{cw}}(A) = \big(\sup_i A_i\big)_{i=1}^{s}.$$

Then for every edge $e = (x,y) \in N$ and every $b \in Nm$,

$$\sup_{1,\mathrm{cw}}\big(bdf_H(e,b)\big) \ \leqslant_{\mathrm{cw}} \ \min\Big\{\sup_{1,\mathrm{cw}}\big(adf_H(x,a)\big),\ \sup_{1,\mathrm{cw}}\big(adf_H(y,a')\big)\Big\}$$

for all $(a,a') \in Ml \times Ml$, in the coordinate-wise order $\leqslant_{\mathrm{cw}}$.

4. *Contradiction constraint.* For all $(b_1,b_2),(c_1,c_2) \in Nm^2$,

$$bCf\big((b_1,b_2),(c_1,c_2)\big) \ \leqslant_{\mathrm{cw}} \ \min\big\{aCf(b_1,c_1),\ aCf(b_2,c_2)\big\}.$$

Definition 3.13 (n-SuperHyperPlithogenic Graph)**.** Let $n \geqslant 1$. Define recursively

$$\tilde{\mathcal{P}}_k\big([0,1]^s\big) = \big\{A \subseteq \tilde{\mathcal{P}}_{k-1}([0,1]^s) : A \neq \varnothing\big\}, \quad k \geqslant 1.$$

An *n-SuperHyperPlithogenic Graph* is

$$PG_{nH} = (PM_{nH},\ PN_{nH}),$$

with

$$PM_{nH} = (M,\ l,\ Ml,\ adf_{nH},\ aCf), \quad PN_{nH} = (N,\ m,\ Nm,\ bdf_{nH},\ bCf),$$

where

- $adf_{nH} : M \times Ml \to \tilde{\mathcal{P}}_n([0,1]^s)$,
- $bdf_{nH} : N \times Nm \to \tilde{\mathcal{P}}_n([0,1]^s)$,

and the same coordinate-wise suprema operator $\sup_{\mathrm{cw}n}$ is defined by

$$\sup_{k,\mathrm{cw}}(A) = \sup_{k-1,\mathrm{cw}}\big(\bigcup A\big), \quad k \geqslant 2.$$

The *appurtenance* and *contradiction* constraints of Definition 3.12 hold with $\sup_{\mathrm{cw}n}$ in place of $\sup_{\mathrm{cw}1}$.

Theorem 3.14 (Reduction at $n = 1$)**.** *If $n = 1$, then PG_{1H} of Definition 3.13 coincides with PG_H of Definition 3.12.*

Proof. By definition $\tilde{\mathcal{P}}_1([0,1]^s) = \tilde{\mathcal{P}}_1([0,1]^s)$. Hence $adf_{1H} = adf_H$ and $bdf_{1H} = bdf_H$, and $\sup_{\mathrm{cw}1}$ is the same in both. All constraints agree, so the two notions coincide. □

Theorem 3.15 (Generalization of HyperNeutrosophic Graph)**.** *Every HyperNeutrosophic Graph $\big(V,E,\tilde{\sigma},\tilde{\mu}\big)$ (Definition 3.5) embeds into a HyperPlithogenic Graph by choosing*

$$Ml = \{T,I,F\}, \quad adf_H(v,T) = \{T' \mid (T',I',F') \in \tilde{\sigma}(v)\},$$

and similarly for I,F, with aCf trivial. Edges map analogously. The coordinate-wise suprema constraint on $\tilde{\sigma},\tilde{\mu}$ becomes the HyperPlithogenic appurtenance constraint on adf_H, bdf_H.

Proof. Given $\tilde{\sigma}(v) \subseteq [0,1]^3$, define for each coordinate $j \in \{T, I, F\}$:

$$adf_H(v,j) = \{\, x_j : (x_T, x_I, x_F) \in \tilde{\sigma}(v) \,\} \subseteq [0,1].$$

Then $\sup_{\mathrm{cw}1}(adf_H(v,j)) = \sup\{x_j\} = \sup^j(\tilde{\sigma}(v))$. Applying the Hyperneutrosophic consistency $\sup^j(\tilde{\mu}(e)) \leqslant \min\{\sup^j(\tilde{\sigma}(u)), \sup^j(\tilde{\sigma}(v))\}$ for each j, we recover exactly

$$\sup_{1}^{\mathrm{cw}}\big(bdf_H(e,j)\big) \leqslant_{\mathrm{cw}} \min\{\sup_{1}^{\mathrm{cw}}(adf_H(u,j)), \sup_{1}^{\mathrm{cw}}(adf_H(v,j))\}.$$

Thus the HyperPlithogenic Graph constraints hold. The contradiction functions aCf, bCf can be taken zero, satisfying symmetry and reflexivity trivially. □

Theorem 3.16 (Embedding into SuperHyperPlithogenic Graph)**.** *Let*

$$G_{nHN} = (V,\ E,\ \tilde{\sigma}_n,\ \tilde{\mu}_n)$$

be an n–SuperHyperneutrosophic Graph (Definition 3.7). Then there is a natural embedding of G_{nHN} into an n–SuperHyperPlithogenic Graph

$$PG_{nH} = (PM_{nH},\ PN_{nH}),$$

obtained by projecting each nested neutrosophic membership set onto its three coordinates at every level.

Proof. We shall construct

$$PM_{nH} = (V,\ l,\ Ml,\ adf_{nH},\ aCf), \quad PN_{nH} = (E,\ m,\ Nm,\ bdf_{nH},\ bCf),$$

where $Ml = \{T, I, F\}$ and $Nm = Ml \times Ml$. Throughout, $\tilde{\mathcal{P}}_k([0,1]^3)$ denotes the family of all nonempty nested sets of depth k whose "points'' lie in $[0,1]^3$.

(1) Vertex side. For each $v \in V$ and each $p \in Ml$, define

$$adf_{nH}(v,p) = \{\, A_p \mid A \in \tilde{\sigma}_n(v) \,\} \subseteq \tilde{\mathcal{P}}_n([0,1]),$$

where $A_p \subseteq [0,1]$ is the "p-coordinate projection'' of the neutrosophic set $A \subseteq [0,1]^3$:

$$A_p = \{\, x_p : (x_T, x_I, x_F) \in A \,\}.$$

Since $\tilde{\sigma}_n(v) \neq \varnothing$, each $adf_{nH}(v,p)$ is nonempty.

We take the same contradiction function $aCf : Ml \times Ml \to [0,1]$ already given in the plithogenic-graph framework (satisfying reflexivity $aCf(p,p) = 0$ and symmetry).

(2) Edge side. For each edge $e \in E$ and each ordered pair $(p,q) \in Nm$, set

$$bdf_{nH}(e,(p,q)) = \begin{cases} \{\, A_p : A \in \tilde{\mu}_n(e) \,\}, & p = q, \\ \{\{0\}\}, & p \neq q, \end{cases}$$

so that $bdf_{nH}(e,(p,p))$ is the projection of the neutrosophic membership sets of e onto the p-coordinate, and all "off-diagonal'' pairs receive the trivial zero–set. Again, $bdf_{nH}(e,(p,q))$ is nonempty in either case. The contradiction function bCf on Nm is defined in the usual plithogenic way, for instance

$$bCf\big((p,p),\,(q,q)\big) = \min\{\, aCf(p,q),\, aCf(p,q)\} = aCf(p,q),$$

and is extended to mixed pairs by symmetry/reflexivity.

(3) Checking the hyper-appurtenance constraint. Fix $e = (u,v) \in E$ and a "diagonal'' attribute (p,p). Let $\sup_n^p$ denote the recursive supremum of the p-coordinate on $\tilde{\mathcal{P}}_n([0,1]^3)$. By the hypothesis of the neutrosophic graph,

$$\sup_{n}^{p}\big(\tilde{\mu}_n(e)\big) \leqslant \min\Big\{\sup_{n}^{p}\big(\tilde{\sigma}_n(u)\big), \sup_{n}^{p}\big(\tilde{\sigma}_n(v)\big)\Big\}.$$

On the plithogenic side, each element of $bdf_{nH}(e,(p,p))$ is some $x \in A_p$ for $A \in \tilde{\mu}_n(e)$. But $\sup A_p = \sup_n^p(A)$, and hence

$$x \leqslant \sup_n^p(A) \leqslant \sup_n^p\big(\tilde{\mu}_n(e)\big) \leqslant \min\big\{\sup_n^p(\tilde{\sigma}_n(u)),\, \sup_n^p(\tilde{\sigma}_n(v))\big\}.$$

On the plithogenic side,

$$\max adf_{nH}(u,p) = \sup_n^p(\tilde{\sigma}_n(u))$$

and similarly for v. Thus

$$\alpha \leqslant \min\{\max adf_{nH}(u,p), \max adf_{nH}(v,p)\}$$

, verifying the edge-appurtenance constraint for the plithogenic graph.

(4) Checking the contradiction constraint. For two "diagonal'' attribute pairs $(p,p),(q,q)$, by construction

$$bCf\big((p,p),(q,q)\big) = aCf(p,q),$$

and the neutrosophic-graph requirement $aCf(p,q) \leqslant \min\{aCf(p,p), aCf(q,q)\} = 0$ immediately implies the plithogenic-graph contradiction constraint. Mixed pairs (p,q) with $p \neq q$ carry only the zero–set and thus pose no further restriction.

Since all vertex- and edge-appurtenance and contradiction axioms of an n–SuperHyperPlithogenic Graph are satisfied, the above coordinate-projection construction is indeed an embedding of G_{nHN} into PG_{nH}. □

4 Result: Uncertain Graph Neural Networks

In this section, we explore uncertain graph networks, including Fuzzy Graph Neural Networks, Neutrosophic Graph Neural Networks, and Plithogenic Graph Neural Networks.

4.1 HyperFuzzy Graph Neural Network

The definitions of the HyperFuzzy Graph Neural Network and its further extension, the SuperHyperFuzzy Graph Neural Network, are provided below.

Definition 4.1 (HyperFuzzy Graph Neural Network (HF-GNN))**.** Let $G = (V,E)$ be a finite simple graph. Fix an integer $M \geqslant 1$. A *HyperFuzzy Graph Neural Network* over G is a quintuple

$$\text{HF-GNN} \;=\; \big(G,\; \tilde{\mathcal{F}}_V,\; \tilde{\mathcal{F}}_E,\; \{R_k\}_{k=1}^K,\; D\big),$$

consisting of:

1. **Hyperfuzzy fuzzification.**

$$\tilde{\mathcal{F}}_V :\; \mathcal{X}_V \;\longrightarrow\; \tilde{P}\big([0,1]^M\big), \qquad \tilde{\mathcal{F}}_E :\; \mathcal{X}_E \;\longrightarrow\; \tilde{P}\big([0,1]^M\big),$$

   where $\mathcal{X}_V$ (resp. $\mathcal{X}_E$) is the set of possible vertex (resp. edge) attributes, and $\tilde{P}([0,1]^M)$ denotes the collection of all nonempty subsets of $[0,1]^M$. Thus each vertex v with feature x_v is assigned a *hyperfuzzy membership set* $\tilde{\mathcal{F}}_V(x_v) \subseteq [0,1]^M$, and similarly for edges.

2. **Rule layer.** A finite collection of *hyperfuzzy rules* $\{R_k\}_{k=1}^K$ is specified. Each rule R_k consists of

   - two fuzzy sets $A_k \subseteq [0,1]^M$ and $B_k \subseteq [0,1]^M$,
   - a trainable aggregation function $f_k : \mathrm{R}^d \times \mathrm{R}^d \to \mathrm{R}^d$,

intended to capture a pattern such as

$$\text{IF } m_V \in A_k \text{ AND } m_E \in B_k \quad \text{THEN} \quad \text{message } f_k(h_v, h_u)\,,$$

where $m_V \in \tilde{\mathcal{F}}_V(x_v)$ and $m_E \in \tilde{\mathcal{F}}_E(x_{u,v})$.

3. **Defuzzification.** A mapping
$$D:\ \mathrm{R}^d \longrightarrow \mathrm{R}^d$$
converts the aggregated real-valued messages into updated node features.

Definition 4.2 (Forward Propagation in HF-GNN)**.** Let HF-GNN $= (G, \tilde{\mathcal{F}}_V, \tilde{\mathcal{F}}_E, \{R_k\}, D)$ be as above. Given initial vertex features $h_v^{(0)} \in \mathrm{R}^d$ and edge features $h_{u,v}^{(0)} \in \mathrm{R}^d$, we inductively define for $l = 1, 2, \ldots, L$:

1. **Hyperfuzzy fuzzification.**
For each $v \in V$,
$$M_v^{(l)} \ = \ \tilde{\mathcal{F}}_V\big(h_v^{(l-1)}\big) \ \subseteq \ [0,1]^M,$$
and for each edge $e = (u, v) \in E$,
$$M_e^{(l)} \ = \ \tilde{\mathcal{F}}_E\big(h_{u,v}^{(l-1)}\big) \ \subseteq \ [0,1]^M.$$

2. **Rule evaluation and firing strengths.**
For each rule $R_k = (A_k, B_k, f_k)$ and each directed edge $u \to v$, define the firing strength
$$r_k^{(l)}(u \to v) \ = \ \max_{m_V \in M_v^{(l)},\, m_E \in M_e^{(l)}} \min\{\,\mathrm{dist}(m_V, A_k), \mathrm{dist}(m_E, B_k)\}\,,$$
where $\mathrm{dist}(m, A) = 1 - \sup_{a \in A}\big(1 - \|m - a\|\big)$ is a suitable degree-of-membership measure. (Any other fuzzy membership operator may be used.)

3. **Normalization.**
For each edge $u \to v$, normalize
$$\hat{r}_k^{(l)}(u \to v) \ = \ \frac{r_k^{(l)}(u \to v)}{\sum_{j=1}^{K} r_j^{(l)}(u \to v)} \quad \text{(if the denominator is positive; else all } \hat{r}_k = 0).$$

4. **Message aggregation.**
The message from u to v at layer l is
$$m_{u \to v}^{(l)} \ = \ \sum_{k=1}^{K} \hat{r}_k^{(l)}(u \to v)\ f_k\big(h_v^{(l-1)},\, h_u^{(l-1)}\big).$$
Then node v aggregates
$$M_v^{(l)} \ = \ \sum_{u \in N(v)} m_{u \to v}^{(l)}.$$

5. **Defuzzification / Update.**
Finally, update the node feature by
$$h_v^{(l)} \ = \ D\big(M_v^{(l)}\big).$$

After L layers, one may apply a task-specific readout (e.g. softmax) to $\{h_v^{(L)}\}$.

Example 4.3 (Customer–Item Recommendation via HF-GNN)**.** Let G be a bipartite graph with
$$V = V_{\text{cust}} \cup V_{\text{item}}, \quad E \subseteq V_{\text{cust}} \times V_{\text{item}}.$$
Each customer c has an embedding $h_c^{(0)} \in \mathrm{R}^d$, and each interaction edge (c, i) has feature $h_{c,i}^{(0)}$. Fix $M = 2$ latent interests (e.g. fashion vs. tech).

- *Hyperfuzzy fuzzification.*

$$M_c^{(1)} \;=\; \big\{\,(0.7,0.5),\,(0.8,0.6)\big\}, \quad M_{c,i}^{(1)} = \big\{\,(0.9,0.3),\,(0.5,0.8)\big\}.$$

- *Two example rules.*

$$R_1 : A_1 = [0.6,1]^2,\; B_1 = [0.6,1]^2,\; f_1(x,y) = W_1[x\|y],$$

$$R_2 : A_2 = [0,0.4]\times[0,1],\; B_2 = [0,0.4]\times[0,1],\; f_2(x,y) = W_2[x\|y].$$

- *Compute firing strengths.* For (c,i):

$$r_1 = \max_{m_V,m_E} \min\{\mathrm{dist}(m_V,A_1),\mathrm{dist}(m_E,B_1)\} \approx 0.7, \quad r_2 \approx 0.3.$$

- *Normalize and aggregate.* $\hat{r}_1 = 0.7/(0.7+0.3) = 0.7$, $\hat{r}_2 = 0.3$. Then

$$m_{c\to i} = 0.7\, f_1(h_c,h_i) + 0.3\, f_2(h_c,h_i), \quad h_i^{(1)} = D\big(\sum_c m_{c\to i}\big).$$

The final $h_i^{(L)}$ yields a crisp recommendation score for each item i.

Theorem 4.4 (HF-GNN Generalizes F-GNN)**.** *Let $\mathcal{F}_V : \mathcal{X}_V \to [0,1]^M$ and $\mathcal{F}_E : \mathcal{X}_E \to [0,1]^M$ be the fuzzification functions of a standard Fuzzy GNN. Then it embeds into an HF-GNN by setting*

$$\tilde{\mathcal{F}}_V(x) = \{\mathcal{F}_V(x)\}, \quad \tilde{\mathcal{F}}_E(x) = \{\mathcal{F}_E(x)\}.$$

Under this identification, all rule–evaluation, normalization, and defuzzification steps coincide with those of the original F-GNN. Hence every F-GNN is a special case of HF-GNN.

Proof. Let $\mathcal{F}_V : \mathcal{X}_V \to [0,1]^M$ and $\mathcal{F}_E : \mathcal{X}_E \to [0,1]^M$ be the fuzzification maps of a standard F-GNN, and let F-GNN $= (G, \mathcal{F}_V, \mathcal{F}_E, \{R_k\}, D)$ denote its quintuple (with the same rule layer $\{R_k\}$ and defuzzifier D). We define an HF-GNN by

$$\tilde{\mathcal{F}}_V(x) \;:=\; \{\mathcal{F}_V(x)\}, \quad \tilde{\mathcal{F}}_E(x) \;:=\; \{\mathcal{F}_E(x)\},$$

so that each hyperfuzzy membership set is a *singleton* in $[0,1]^M$.

We check that under this identification, every step of the HF-GNN forward pass (Definition 4.2) reduces exactly to the corresponding step of the F-GNN:

1. Fuzzification layer. For a vertex v with feature x_v,

$$M_v = \tilde{\mathcal{F}}_V(x_v) = \{\mathcal{F}_V(x_v)\},$$

and similarly for an edge $e = (u,v)$, $M_e = \{\mathcal{F}_E(x_e)\}$. Thus in every maximization or iteration over $m_V \in M_v$ (resp. $m_E \in M_e$), there is only one choice:

$$m_V = \mathcal{F}_V(x_v), \quad m_E = \mathcal{F}_E(x_e).$$

2. Rule evaluation. Recall that in HF-GNN one sets for each rule $R_k = (A_k, B_k, f_k)$,

$$r_k(u\to v) = \max_{m_V\in M_v,\; m_E\in M_e} \min\big\{\mathrm{dist}(m_V,A_k),\,\mathrm{dist}(m_E,B_k)\big\}.$$

Since $|M_v| = |M_e| = 1$, this becomes

$$r_k(u\to v) = \min\big\{\mathrm{dist}\big(\mathcal{F}_V(x_v),A_k\big),\,\mathrm{dist}\big(\mathcal{F}_E(x_{u,v}),B_k\big)\big\},$$

which is exactly the firing strength computed in the original F-GNN for rule k.

3. Normalization. In both HF-GNN and F-GNN, one defines

$$\hat{r}_k(u \to v) = \frac{r_k(u \to v)}{\sum_{j=1}^{K} r_j(u \to v)},$$

so the normalization layer is identical.

4. Message aggregation and update. Both models compute

$$m_{u \to v} = \sum_{k=1}^{K} \hat{r}_k(u \to v)\ f_k\big(h_v, h_u\big),$$

and then aggregate over neighbors and apply the same defuzzifier D:

$$h'_v \ = \ D\Big(\sum_{u \in N(v)} m_{u \to v}\Big).$$

Since the messages $m_{u \to v}$ coincide layer by layer, the node features $h_v^{(l)}$ also coincide exactly.

At every stage—fuzzification, rule evaluation, normalization, aggregation, and defuzzification—the HF-GNN with singleton hyperfuzzy sets reproduces precisely the computations of the original F-GNN. Therefore, every Fuzzy GNN is realized as a special case of the HyperFuzzy GNN by restricting each hyperfuzzy membership set to a singleton. □

4.2 HyperNeutrosophic Graph Neural Network (HN-GNN)

We now introduce a HyperNeutrosophic Graph Neural Network, which extends the standard Neutrosophic GNN by allowing each vertex and edge to carry a *set* of neutrosophic membership triplets rather than a single one.

Definition 4.5 (HyperNeutrosophic Graph Neural Network)**.** A *HyperNeutrosophic Graph Neural Network* is a quintuple

$$\text{HN-GNN} \ = \ \big(G,\ \tilde{\mathcal{N}}_V,\ \tilde{\mathcal{N}}_E,\ \{R_k\}_{k=1}^{K},\ D\big),$$

where:

1. $G = (V, E)$ is a finite directed or undirected graph.
2. $\tilde{\mathcal{N}}_V : \mathcal{X}_V \to \tilde{\mathcal{P}}\big([0,1]^3\big)$ is the *vertex fuzzification map*. Here

   $$\tilde{\mathcal{P}}([0,1]^3) \ = \ \{\, A \subseteq [0,1]^3 : A \neq \varnothing\},$$

   and for each vertex attribute $x_v \in \mathcal{X}_V$, the set $\tilde{\mathcal{N}}_V(x_v) \subseteq [0,1]^3$ consists of one or more neutrosophic triplets (T, I, F) satisfying $0 \leqslant T + I + F \leqslant 3$.
3. $\tilde{\mathcal{N}}_E : \mathcal{X}_E \to \tilde{\mathcal{P}}\big([0,1]^3\big)$ is the *edge fuzzification map*, defined analogously on edge-attribute space $\mathcal{X}_E$.
4. For $k = 1, \ldots, K$, each *rule* R_k consists of:
   - Two *antecedent sets* $A_k, B_k \subseteq [0,1]^3$, each nonempty.
   - A *consequent function* $g_k : [0,1]^3 \times [0,1]^3 \to \mathbb{R}^d$.
5. $D : \mathbb{R}^d \to \mathbb{R}^d$ is a *defuzzification or read-out map*, typically a normalization or softmax.

Definition 4.6 (Forward Pass of HN-GNN)**.** Let HN-GNN $= (G, \tilde{\mathcal{N}}_V, \tilde{\mathcal{N}}_E, \{R_k\}, D)$ be as above, and suppose each vertex v carries an initial feature $h_v^{(0)}$. We compute successive layers $h_v^{(l)} \in \mathbb{R}^d$ by:

1. **Fuzzification.** For each vertex v and each neighbor u with edge $e = (u, v)$,
$$M_v \;=\; \tilde{\mathcal{N}}_V(h_v^{(l-1)}), \quad M_u \;=\; \tilde{\mathcal{N}}_V(h_u^{(l-1)}), \quad M_e \;=\; \tilde{\mathcal{N}}_E\big(x_e\big),$$
each a nonempty subset of $[0,1]^3$.

2. **Rule activation.** For each rule $R_k = (A_k, B_k, g_k)$, define its *firing strength* on the triple $(u \to v)$ by
$$r_k^{(l)}(u \to v) \;=\; \max_{\substack{t_v \in M_v \\ t_u \in M_u \\ t_e \in M_e}} \min\big\{\, \delta\big(t_v, A_k\big),\; \delta\big(t_u, B_k\big)\big\},$$
where $\delta(t, A) = 1 - \mathrm{dist}(t, A)$ is a normalized membership degree, and dist any metric on $[0,1]^3$.

3. **Normalization.** Set
$$\hat{r}_k^{(l)}(u \to v) \;=\; \frac{r_k^{(l)}(u \to v)}{\sum_{j=1}^{K} r_j^{(l)}(u \to v)}.$$

4. **Message computation.** Compute the message from u to v by
$$m_{u \to v}^{(l)} \;=\; \sum_{k=1}^{K} \hat{r}_k^{(l)}(u \to v)\; g_k\big(t_v^*,\, t_u^*\big),$$
where (t_v^*, t_u^*) realize the maximum in step 2.

5. **Vertex update.** Aggregate incoming messages and apply a learnable transform Φ:
$$h_v^{(l)} \;=\; \sigma\Big(\Phi\big(h_v^{(l-1)},\, \sum_{u \in N(v)} m_{u \to v}^{(l)}\big)\Big),$$
where σ is a nonlinear activation (e.g. ReLU).

6. **Defuzzification / Readout.** After L layers, produce final outputs by $y_v = D\big(h_v^{(L)}\big)$.

Example 4.7 (HN-GNN on E-Commerce Product Graph)**.** Let $G = (V, E)$ be the graph whose vertices are products and whose edge (p, q) indicates frequent co-purchase. Customer reviews induce for each product p a *hyperneutrosophic membership set*
$$\tilde{\mathcal{N}}_V(x_p) \;=\; \{\,(0.7, 0.1, 0.2),\; (0.4, 0.5, 0.1)\},$$
where each (T, I, F) encodes positive/indeterminate/negative sentiments. Similarly, each edge carries $\tilde{\mathcal{N}}_E(x_{p,q}) \subseteq [0,1]^3$ describing uncertain correlation strength. A rule R_k might fire strongly only when both T-degrees are high, and produce a cross-sell score $g_k(T_p, T_q) \in \mathbb{R}$. By aggregating such rules, normalizing, and updating via Definition 4.6, the HN-GNN yields a crisp bundling recommendation $y_{p,q} \in \mathbb{R}$ that robustly accounts for conflicting or ambiguous feedback.

Theorem 4.8 (HN-GNN Generalizes N-GNN)**.** *Every standard Neutrosophic Graph Neural Network, which assigns exactly one $(T, I, F) \in [0,1]^3$ to each vertex and edge, can be embedded into an HN-GNN by treating each singleton as a one-element hyperneutrosophic set. All rule-evaluation, normalization, and update equations then coincide.*

Proof. Let
$$\mathcal{N}_V : \mathcal{X}_V \longrightarrow [0,1]^3, \qquad \mathcal{N}_E : \mathcal{X}_E \longrightarrow [0,1]^3$$
be the fuzzification maps of the original Neutrosophic GNN (N-GNN), so that for each vertex v and edge e,
$$\mathcal{N}_V(x_v) = (T_v, I_v, F_v), \quad \mathcal{N}_E(x_e) = (T_e, I_e, F_e).$$
We now view this N-GNN as an HN-GNN by setting
$$\tilde{\mathcal{N}}_V(x_v) \;=\; \{\mathcal{N}_V(x_v)\}, \qquad \tilde{\mathcal{N}}_E(x_e) \;=\; \{\mathcal{N}_E(x_e)\}.$$
Thus each "hyperneutrosophic" set is a singleton.

We check that at every step of the HN-GNN forward pass (Definition 4.6), the computations reduce exactly to those of the N-GNN:

1. **Fuzzification.** For each vertex v,
$$M_v \;=\; \tilde{\mathcal{N}}_V(x_v) \;=\; \{\,(T_v, I_v, F_v)\},$$
and similarly $M_e = \{(T_e, I_e, F_e)\}$ for each edge.

2. **Rule activation.** Recall in the HN-GNN we define for each rule $R_k = (A_k, B_k, g_k)$ and each directed edge $(u \to v)$:
$$r_k(u \to v) \;=\; \max_{\substack{t_v \in M_v \\ t_u \in M_u \\ t_e \in M_e}} \min\{\,\delta(t_v, A_k),\, \delta(t_u, B_k)\},$$
where $\delta(t, A) = 1 - \mathrm{dist}(t, A)$. Since M_v and M_u and M_e are singletons, the only triple to consider is $t_v = (T_v, I_v, F_v)$ and $t_u = (T_u, I_u, F_u)$. Hence
$$r_k(u \to v) = \min\big\{\delta\big((T_v, I_v, F_v), A_k\big), \delta\big((T_u, I_u, F_u), B_k\big)\big\},$$
which is exactly the firing strength in the original N-GNN.

3. **Normalization.** In HN-GNN we set $\hat{r}_k(u \to v) = r_k(u \to v) / \sum_j r_j(u \to v)$, identical to the N-GNN normalization since the r_k agree.

4. **Message computation and update.** The HN-GNN message $m_{u \to v} = \sum_k \hat{r}_k\, g_k(t_v^*, t_u^*)$, where (t_v^*, t_u^*) realize the maximum in step 2, reduces to $\sum_k \hat{r}_k\, g_k\big((T_v, I_v, F_v), (T_u, I_u, F_u)\big)$, again matching the N-GNN's message-passing rule. The subsequent vertex update
$$h_v \;=\; \sigma\big(\Phi(h_v, \sum_u m_{u \to v})\big)$$
is unchanged.

5. **Defuzzification.** Finally, the read-out $y_v = D(h_v)$ coincides in both models.

Since every step—rule evaluation, normalization, message aggregation, vertex update and read-out—yields the same formulas under this singleton embedding, the standard N-GNN is recovered exactly. Therefore, every Neutrosophic GNN is a special case of the HyperNeutrosophic GNN in which each hyperneutrosophic set contains exactly one triplet. □

4.3 HyperPlithogenic Graph Neural Network (HP-GNN)

The definitions of the HyperPlithogenic Graph Neural Network and its further extension, the SuperHyperPlithogenic Graph Neural Network, are provided below.

Definition 4.9 (HyperPlithogenic Graph Neural Network (HP-GNN))**.** A *HyperPlithogenic Graph Neural Network* is a quintuple
$$\text{HP-GNN} \;=\; \big(G,\, \tilde{\Phi}_V,\, \tilde{\Phi}_E,\, \mathcal{R}_P,\, \mathcal{D}_P\big),$$
where:

- $G = (V, E)$ is a (simple, finite) graph.
- $\tilde{\Phi}_V : \mathcal{X}_V \to \widetilde{\mathcal{P}}\big([0,1]^s\big)$ is the *hyperplithogenic vertex fuzzification function*. For each vertex attribute $x_v \in \mathcal{X}_V$,
$$\tilde{\Phi}_V(x_v) \;\subseteq\; [0,1]^s, \quad \tilde{\Phi}_V(x_v) \neq \varnothing,$$
is a nonempty set of membership vectors $\alpha = (\alpha_1, \ldots, \alpha_s)$ encoding degrees of appurtenance to s plithogenic attribute values.
- $\tilde{\Phi}_E : \mathcal{X}_E \to \widetilde{\mathcal{P}}\big([0,1]^s\big)$ is the *hyperplithogenic edge fuzzification function*, assigning to each edge attribute $x_e \in \mathcal{X}_E$ a nonempty set $\tilde{\Phi}_E(x_e) \subseteq [0,1]^s$.

- $\mathcal{R}_P = \{R_1, \ldots, R_K\}$ is a finite collection of *hyperplithogenic rules*. Each rule R_k consists of:

$$R_k = (A_{v,k},\ A_{u,k},\ A_{e,k},\ f_k),$$

where $A_{v,k}, A_{u,k}, A_{e,k} \subseteq [0,1]^s$ are (crisp) subsets serving as antecedents for the source-vertex, target-vertex, and edge, and $f_k : [0,1]^s \times [0,1]^s \times [0,1]^s \to \mathbb{R}^d$ is a trainable *consequence* function.

- $\mathcal{D}_P$ is the *defuzzification* map that aggregates the rule outputs into a final crisp embedding or label.

Definition 4.10 (Forward Pass in HP-GNN)**.** Let $G = (V, E)$ be an input graph with vertex attributes $x_v \in \mathcal{X}_V$ and edge attributes $x_{u,v} \in \mathcal{X}_E$. The HP-GNN computes messages and updates in four phases:

1. **Hyperplithogenic Fuzzification.**
   For each vertex $v \in V$, compute its membership set

$$M_v \ = \ \tilde{\Phi}_V(x_v) \ \subseteq \ [0,1]^s, \quad M_v \neq \varnothing.$$

   For each edge $(u, v) \in E$, compute

$$M_{u,v} = \tilde{\Phi}_E(x_{u,v}) \ \subseteq \ [0,1]^s, \quad M_{u,v} \neq \varnothing.$$

2. **Rule Firing.**
   For each directed edge $(u \to v)$ and each rule $R_k = (A_{v,k}, A_{u,k}, A_{e,k}, f_k)$, define its *firing strength*

$$r_k(u \to v) = \max_{\substack{\alpha_v \in M_v \\ \alpha_u \in M_u \\ \alpha_e \in M_{u,v}}} \min\big\{\mathbf{1}_{A_{v,k}}(\alpha_v),\ \mathbf{1}_{A_{u,k}}(\alpha_u),\ \mathbf{1}_{A_{e,k}}(\alpha_e)\big\},$$

   where $\mathbf{1}_A(x) = 1$ if $x \in A$, else 0. Intuitively, one searches for a triple $(\alpha_v, \alpha_u, \alpha_e)$ satisfying all antecedents.

3. **Normalization.**
   Normalize the firing strengths:

$$\hat{r}_k(u \to v) \ = \ \frac{r_k(u \to v)}{\sum_{j=1}^{K} r_j(u \to v)} \quad \text{if } \sum_j r_j > 0,$$

   and set $\hat{r}_k = 0$ if all $r_j = 0$.

4. **Message & Update.**
   Compute the message from u to v as

$$m_{u \to v} = \sum_{k=1}^{K} \hat{r}_k(u \to v)\, f_k\big(\alpha_v^*,\ \alpha_u^*,\ \alpha_e^*\big),$$

   where $(\alpha_v^*, \alpha_u^*, \alpha_e^*)$ is any maximizer in step 2. Then update each vertex embedding:

$$h_v \ = \ \sigma\Big(W_1\, h_v \ + \ W_2 \sum_{u:(u,v)\in E} m_{u \to v}\Big),$$

   with trainable matrices W_1, W_2 and nonlinearity σ.

Example 4.11 (HP-GNN for Contradictory Product Attributes)**.** In a marketing network G, vertices v are customers and edges (v, u) encode social influence. Each customer has plithogenic attributes (brand loyalty, price sensitivity, aesthetics), so

$$\tilde{\Phi}_V(x_v) = \{\,(0.8, 0.3, 0.5),\ (0.4, 0.6, 0.7)\},$$

reflecting two plausible preference profiles in $[0,1]^3$. Each edge (v,u) also has a set

$$\tilde{\Phi}_E(x_{v,u}) = \{\,(0.9,0.1,0.2),\ (0.5,0.7,0.4)\},$$

modeling possibly conflicting measures of influence.

We define three rules:

$$\begin{aligned}
R_1 : A_{v,1} &= \{\alpha : \alpha_1 \geqslant 0.7\},\ A_{u,1} = \{\alpha : \alpha_1 \geqslant 0.7\},\ A_{e,1} = \{\alpha : \alpha_1 \geqslant 0.5\},\\
R_2 : A_{v,2} &= \{\alpha : \alpha_2 \geqslant 0.5\},\ A_{u,2} = \{\alpha : \alpha_2 \geqslant 0.5\},\ A_{e,2} = \{\alpha : \alpha_2 \geqslant 0.4\},\\
R_3 : A_{v,3} &= \{\alpha : \alpha_3 \geqslant 0.6\},\ A_{u,3} = \{\alpha : \alpha_3 \geqslant 0.6\},\ A_{e,3} = \{\alpha : \alpha_3 \geqslant 0.3\},
\end{aligned}$$

with simple linear consequences $f_k(\alpha_v, \alpha_u, \alpha_e) = \alpha_v + \alpha_u + \alpha_e$. The HP-GNN aggregates these scenario-specific rule-based messages and updates each customer's embedding, capturing contradictory or uncertain attribute interactions more richly than a classical GNN or single-valued plithogenic GNN.

Theorem 4.12 (HP-GNN Generalizes P-GNN)**.** *Let $\mathcal{P}_V : \mathcal{X}_V \to [0,1]^s$ and $\mathcal{P}_E : \mathcal{X}_E \to [0,1]^s$ be the vertex/edge fuzzifiers of a standard Plithogenic GNN (P-GNN). Then it embeds into an HP-GNN by*

$$\tilde{\Phi}_V(x) = \{\mathcal{P}_V(x)\}, \quad \tilde{\Phi}_E(x) = \{\mathcal{P}_E(x)\}.$$

Under this identification, all rule-firing, normalization, message and update formulas coincide with those of the original P-GNN. Hence every P-GNN is a special case of HP-GNN.

Proof. Let $\mathcal{P}_V : \mathcal{X}_V \to [0,1]^s$ and $\mathcal{P}_E : \mathcal{X}_E \to [0,1]^s$ be the fuzzification maps of a Plithogenic GNN (P-GNN). We define an embedding into the HyperPlithogenic GNN (HP-GNN) by

$$\tilde{\Phi}_V(x) \ :=\ \big\{\mathcal{P}_V(x)\big\}, \quad \tilde{\Phi}_E(x) \ :=\ \big\{\mathcal{P}_E(x)\big\}.$$

Thus for each vertex v and edge (u,v), their hyperplithogenic membership sets are singletons:

$$M_v = \tilde{\Phi}_V(x_v) = \{\alpha_v\}, \quad M_{u,v} = \tilde{\Phi}_E(x_{u,v}) = \{\alpha_{u,v}\},$$

where $\alpha_v = \mathcal{P}_V(x_v)$ and $\alpha_{u,v} = \mathcal{P}_E(x_{u,v})$.

We now check that each step of the HP-GNN forward pass reduces exactly to the corresponding P-GNN computation:

1. Rule Firing. In HP-GNN (Definition 4.10), for each rule $R_k = (A_{v,k}, A_{u,k}, A_{e,k}, f_k)$ and edge $(u \to v)$,

$$r_k(u \to v) = \max_{\substack{\alpha'_v \in M_v \\ \alpha'_u \in M_u \\ \alpha'_e \in M_{u,v}}} \min\big\{\mathbf{1}_{A_{v,k}}(\alpha'_v), \mathbf{1}_{A_{u,k}}(\alpha'_u), \mathbf{1}_{A_{e,k}}(\alpha'_e)\big\}.$$

Since $M_v = \{\alpha_v\}$, $M_u = \{\alpha_u\}$, $M_{u,v} = \{\alpha_{u,v}\}$ are singletons, the maximum over those sets is attained at the unique triple $(\alpha_v, \alpha_u, \alpha_{u,v})$. Hence

$$r_k(u \to v) = \min\big\{\mathbf{1}_{A_{v,k}}(\alpha_v), \mathbf{1}_{A_{u,k}}(\alpha_u), \mathbf{1}_{A_{e,k}}(\alpha_{u,v})\big\},$$

which is exactly the firing strength computed by the P-GNN, since in P-GNN one tests membership of $\alpha_v, \alpha_u, \alpha_{u,v}$ against the same antecedent sets.

2. Normalization. Both models normalize the raw firing strengths identically:

$$\hat{r}_k = \frac{r_k}{\sum_{j=1}^{K} r_j},$$

so the normalized weights $\hat{r}_k$ coincide.

3. Message Aggregation. In HP-GNN, the message along $(u \to v)$ is

$$m_{u\to v} = \sum_{k=1}^{K} \hat{r}_k(u \to v)\, f_k\big(\alpha_v, \alpha_u, \alpha_{u,v}\big),$$

where the unique maximizer yields the same arguments to f_k as in P-GNN. Thus $m_{u\to v}$ matches the P-GNN message.

4. Vertex Update. Finally, both networks update each vertex embedding by the same formula (up to notation),

$$h_v \;=\; \sigma\Big(W_1 h_v + W_2 \sum_{u:(u,v)\in E} m_{u\to v}\Big),$$

so the HP-GNN update coincides term-by-term with that of the P-GNN.

Since every step—fuzzification, rule firing, normalization, message and update—is identical under the identification of singleton hyperplithogenic sets with ordinary plithogenic vectors, we conclude that any P-GNN is realized as the special case of an HP-GNN in which each membership set has cardinality one. □

4.4 n-SuperHyperFuzzy Graph Neural Network (n-SHF-GNN)

An n-SuperHyperFuzzy Graph Neural Network (n-SHF-GNN) integrates n-level superhyperfuzzy fuzzification for vertices and edges with hyperfuzzy rules and defuzzification mechanisms.

Definition 4.13 (n-SuperHyperFuzzy Graph Neural Network)**.** Let $n \geqslant 1$. An *n-SuperHyperFuzzy Graph Neural Network* (n-SHF-GNN) is a quintuple

$$\text{n-SHF-GNN} \;=\; \big(G,\, \tilde{\mathcal{F}}_V^{(n)},\, \tilde{\mathcal{F}}_E^{(n)},\, \mathcal{R},\, \mathcal{D}\big),$$

where

1. $G = (V, E)$ is a (finite) graph.
2. $\tilde{\mathcal{F}}_V^{(n)} : \mathcal{X}_V \;\longrightarrow\; \tilde{\mathcal{P}}_n\big([0,1]^M\big)$ is the *n-superhyperfuzzy vertex fuzzifier*. For each $x_v \in \mathcal{X}_V$,

$$\tilde{\mathcal{F}}_V^{(n)}(x_v) \;\subseteq\; \tilde{\mathcal{P}}_n\big([0,1]^M\big), \quad \tilde{\mathcal{F}}_V^{(n)}(x_v) \neq \varnothing,$$

   where $\tilde{\mathcal{P}}_1([0,1]^M)$ is the family of all nonempty subsets of $[0,1]^M$, and recursively $\tilde{\mathcal{P}}_{k+1}([0,1]^M)$ is the family of all nonempty subsets of $\tilde{\mathcal{P}}_k([0,1]^M)$.
3. $\tilde{\mathcal{F}}_E^{(n)} : \mathcal{X}_E \;\longrightarrow\; \tilde{\mathcal{P}}_n\big([0,1]^M\big)$ is the *n-superhyperfuzzy edge fuzzifier*, defined analogously on edge attributes.
4. $\mathcal{R} = \{\mathcal{R}_1, \ldots, \mathcal{R}_K\}$ is a finite collection of *superhyperfuzzy rules*. Each rule $\mathcal{R}_k$ consists of:
   - A collection of n-level antecedent sets

$$A_{v,k} \;\subseteq\; \tilde{\mathcal{P}}_n([0,1]^M), \quad A_{u,k} \;\subseteq\; \tilde{\mathcal{P}}_n([0,1]^M), \quad A_{e,k} \;\subseteq\; \tilde{\mathcal{P}}_n([0,1]^M),$$

     to be tested against the fuzzified vertex–edge memberships.
   - A *consequent* function $f_k : [0,1]^M \times [0,1]^M \times [0,1]^M \to \mathrm{R}^d$, used to compute a partial message or feature update.
5. $\mathcal{D}$ is a *defuzzification* mechanism that aggregates the normalized rule outputs into a crisp vector in R^d.

Definition 4.14 (Forward Pass of n-SHF-GNN)**.** Given input graph $G = (V, E)$ with features $X_V \subseteq \mathcal{X}_V$ and $X_E \subseteq \mathcal{X}_E$, the n-SHF-GNN computes:

(F1) ***n*-SuperHyperFuzzy Fuzzification:**
For each vertex v, compute

$$M_v \;=\; \tilde{\mathcal{F}}_V^{(n)}(x_v) \;\subseteq\; \tilde{\mathcal{P}}_n([0,1]^M]),$$

a nonempty n-nested family of membership vectors, and similarly $M_{u,v} = \tilde{\mathcal{F}}_E^{(n)}(x_{uv})$ for each edge (u,v).

(R1) Rule Evaluation:
For each edge $(u \to v) \in E$ and each rule $\mathcal{R}_k$, define its *firing strength*

$$r_k(u \to v) \;=\; \sup_{\substack{a\in M_v\\ b\in M_u\\ c\in M_{u,v}}} \min\big\{\mathbf{1}_{A_{v,k}}(a), \mathbf{1}_{A_{u,k}}(b), \mathbf{1}_{A_{e,k}}(c)\big\},$$

where $\mathbf{1}_A(\cdot)$ is the indicator of membership in the antecedent set $A \subseteq \tilde{\mathcal{P}}_n([0,1]^M])$.

(N1) Normalization:
Normalize the strengths:

$$\hat{r}_k(u \to v) = \frac{r_k(u \to v)}{\sum_{j=1}^{K} r_j(u \to v)}, \quad \sum_{k=1}^{K} \hat{r}_k(u \to v) = 1.$$

(M1) Message Computation:
Compute the message sent from u to v by

$$m_{u\to v} = \sum_{k=1}^{K} \hat{r}_k(u \to v)\, f_k\big(a^*, b^*, c^*\big),$$

where (a^*, b^*, c^*) is any maximizer in the sup–min in (R1).

(U1) Vertex Update:
Update each vertex embedding $h_v \in \mathrm{R}^d$ by

$$h_v \;=\; \sigma\Big(W_0\, h_v^{\text{old}} \;+\; W_1 \sum_{u:(u,v)\in E} m_{u\to v}\Big),$$

where σ is a nonlinear activation and W_0, W_1 are trainable weights.

(D1) Defuzzification / Readout:
After L layers, optionally apply

$$y = \mathrm{Readout}\big(h_v^{(L)} : v \in V\big)$$

to produce graph- or node-level predictions.

Example 4.15 (3-SHF-GNN for Medical Diagnosis under Multi-Level Uncertainty)**.** Let $G = (V, E)$ be a patient-symptom network with two patients $V = \{v_1, v_2\}$ and a single edge $E = \{(v_1, v_2)\}$ indicating similar symptom profiles. We take $M = 3$ test types: blood glucose (g), cholesterol (c), and MRI severity (m). We set $n = 3$ nesting levels.

Level 1 (Single-Test Fuzzification). For each patient v_i, the fuzzifier

$$\tilde{\mathcal{F}}_V^{(1)}(x_{v_i}) \;\subseteq\; [0,1]^3$$

returns a nonempty set of triples (μ_g, μ_c, μ_m). For example:

$$\tilde{\mathcal{F}}_V^{(1)}(x_{v_1}) = \big\{(0.8, 0.1, 0.1),\ (0.7, 0.2, 0.1)\big\}, \quad \tilde{\mathcal{F}}_V^{(1)}(x_{v_2}) = \big\{(0.6, 0.3, 0.1)\big\}.$$

Here $\mu_g = 0.8$ means "very likely normal glucose," $\mu_c = 0.1$ "unlikely normal cholesterol," etc.

Level 2 (Panel-Level Aggregation). We group the three tests into two panels:

$$A_{v_i} = \tilde{\mathcal{F}}_V^{(2)}(x_{v_i}) \subseteq \mathcal{P}\big(\tilde{\mathcal{F}}_V^{(1)}(x_{v_i})\big).$$

For instance,

$$A_{v_1} = \big\{\{(0.8, 0.1, 0.1)\},\ \{(0.7, 0.2, 0.1), (0.8, 0.1, 0.1)\}\big\}, \quad A_{v_2} = \big\{\{(0.6, 0.3, 0.1)\}\big\}.$$

Each element of A_{v_i} is a choice of one or more level-1 fuzzifications, modeling "blood + cholesterol" vs. "all three tests" panels.

Level 3 (Multi-Panel Correlation). Finally,

$$\tilde{\mathcal{F}}_V^{(3)}(x_{v_i}) \subseteq \mathcal{P}\big(A_{v_i}\big),$$

e.g. for v_1:

$$\tilde{\mathcal{F}}_V^{(3)}(x_{v_1}) = \big\{A_{v_1},\ \{\{(0.7, 0.2, 0.1), (0.8, 0.1, 0.1)\}\}\big\},$$

where the first element is the full set of panels, the second is just the "all-three" panel. Similarly for v_2.

Edge Fuzzification. On the edge (v_1, v_2), we capture symptom co-occurrence uncertainty:

$$\tilde{\mathcal{F}}_E^{(1)}(x_{v_1v_2}) = \big\{(0.5, 0.4, 0.1)\big\}, \quad \tilde{\mathcal{F}}_E^{(2)}(x_{v_1v_2}) = \big\{\{(0.5, 0.4, 0.1)\}\big\}, \quad \tilde{\mathcal{F}}_E^{(3)}(x_{v_1v_2}) = \big\{\{\{(0.5, 0.4, 0.1)\}\}\big\}.$$

SuperHyperfuzzy Rule Example. Define one rule $\mathcal{R}_1$ with antecedents

$$A_{v,1} = \tilde{\mathcal{F}}_V^{(3)}(x_{v_1}), \quad A_{u,1} = \tilde{\mathcal{F}}_V^{(3)}(x_{v_2}), \quad A_{e,1} = \tilde{\mathcal{F}}_E^{(3)}(x_{v_1v_2}),$$

and consequent

$$f_1\big((\mu_g, \mu_c, \mu_m)_{v_1}, (\mu_g, \mu_c, \mu_m)_{v_2}, (\mu_g, \mu_c, \mu_m)_e\big) = \begin{pmatrix} \mu_g \cdot \mu_g', \\ \mu_c \cdot \mu_c', \\ \mu_m \cdot \mu_m' \end{pmatrix} \in \mathrm{R}^3,$$

where $(\mu_g', \mu_c', \mu_m')_{v_2} \in A_{u,1}$ and similarly for the edge.

Forward Pass.

(1) *Fuzzification* (levels 1–3) yields the nested sets above.

(2) *Rule firing strength*

$$r_1(v_1 \to v_2) = \sup_{a \in \tilde{\mathcal{F}}_V^{(3)}(x_{v_1}),\, b \in \tilde{\mathcal{F}}_V^{(3)}(x_{v_2}),\, c \in \tilde{\mathcal{F}}_E^{(3)}(x_{v_1v_2})} \min\{1_A(a), 1_A(b), 1_A(c)\} = 1.$$

(3) *Normalization:* $\hat{r}_1 = 1$.

(4) *Message:*

$$m_{v_1 \to v_2} = f_1\big((0.8, 0.1, 0.1), (0.6, 0.3, 0.1), (0.5, 0.4, 0.1)\big) = (0.4,\ 0.03,\ 0.01).$$

(5) *Update:*

$$h_{v_2} = \sigma\big(W_0 h_{v_2}^{\text{old}} + W_1\, m_{v_1 \to v_2}\big).$$

(6) *Readout:* After L layers, applies a final MLP to $\{h_{v_i}\}$.

This example shows how an n-SHF-GNN can carry uncertainty from individual lab tests up through panel-level and cross-panel reasoning to produce a crisp diagnostic feature.

Theorem 4.16. *If $n = 1$, then $\tilde{\mathcal{P}}_1([0,1]^M) = \tilde{P}([0,1]^M)$, and the definition of n-SHF-GNN collapses exactly to that of the HyperFuzzy GNN (HF-GNN) in Definition 4.1. In particular,*

$$\text{1-SHF-GNN} \;\equiv\; \text{HF-GNN}.$$

Proof. First, note that by definition

$$\tilde{\mathcal{P}}_1\big([0,1]^M\big) \;=\; \big\{\, S \subseteq [0,1]^M \mid S \neq \varnothing \big\} \;=\; \tilde{P}\big([0,1]^M\big),$$

the family of all nonempty subsets of $[0,1]^M$. Hence for $n = 1$ our fuzzification maps

$$\tilde{\mathcal{F}}_V^{(1)} : \mathcal{X}_V \to \tilde{\mathcal{P}}_1([0,1]^M) \quad \text{and} \quad \tilde{\mathcal{F}}_E^{(1)} : \mathcal{X}_E \to \tilde{\mathcal{P}}_1([0,1]^M)$$

take exactly the same range as the hyperfuzzy fuzzifiers in Definition 4.1: $\tilde{\mathcal{F}}_V, \tilde{\mathcal{F}}_E : \mathcal{X} \to \tilde{P}([0,1]^M)$.

Next, consider the *rule evaluation* step. In the 1-SHF-GNN, for each edge (u, v) and rule $\mathcal{R}_k$ we compute

$$r_k(u \to v) = \sup_{a \in \tilde{\mathcal{F}}_V^{(1)}(x_v),\, b \in \tilde{\mathcal{F}}_V^{(1)}(x_u),\, c \in \tilde{\mathcal{F}}_E^{(1)}(x_{u,v})} \min\{\mathbf{1}_{A_{v,k}}(a), \mathbf{1}_{A_{u,k}}(b), \mathbf{1}_{A_{e,k}}(c)\}.$$

Since $\tilde{\mathcal{P}}_1([0,1]^M) = \tilde{P}([0,1]^M)$, these are exactly the sup–min firing strengths used in the HF-GNN. The subsequent normalization $\hat{r}_k = r_k / \sum_j r_j$, message aggregation $\sum_k \hat{r}_k\, f_k(\cdot)$, and defuzzification steps follow identically, with no additional nesting levels to consider.

Finally, the vertex-update and readout equations depend only on these messages $m_{u \to v}$ and on standard GNN aggregation and activation functions, which are unchanged by the value of n.

Therefore, every operation in the 1-SHF-GNN—fuzzification, rule evaluation, normalization, message passing, and defuzzification—coincides exactly with the corresponding operation in the HyperFuzzy GNN. This completes the proof that

$$\text{1-SHF-GNN} \;\equiv\; \text{HF-GNN}.$$

□

4.5 n-SuperHyperNeutrosophic Graph Neural Network (n-SHN-GNN)

An n-SuperHyperNeutrosophic Graph Neural Network (n-SHN-GNN) models uncertainty using n-level nested neutrosophic triplets (T, I, F) for vertices and edges.

Definition 4.17 (n-SuperHyperNeutrosophic GNN (n-SHN-GNN))**.** Let $n \geqslant 1$. An *n-SuperHyperNeutrosophic Graph Neural Network* is a quintuple

$$\text{n-SHN-GNN} \;=\; \big(G,\, \tilde{\mathcal{N}}_V^{(n)},\, \tilde{\mathcal{N}}_E^{(n)},\, \mathcal{R}_N,\, \mathcal{D}_N\big),$$

where:

1. $G = (V, E)$ is a finite simple graph.
2. $\tilde{\mathcal{N}}_V^{(n)} : \mathcal{X}_V \to \tilde{\mathcal{P}}_n([0,1]^3)$ is the *vertex fuzzification*, assigning to each vertex attribute $x_v \in \mathcal{X}_V$ a nonempty n-nested family

   $$\tilde{\mathcal{N}}_V^{(n)}(x_v) \;\subseteq\; \tilde{\mathcal{P}}_n\big([0,1]^3\big), \quad \tilde{\mathcal{N}}_V^{(n)}(x_v) \neq \varnothing.$$

   Here $\tilde{\mathcal{P}}_1([0,1]^3)$ is the set of all nonempty subsets of $[0,1]^3$, and $\tilde{\mathcal{P}}_{k+1}([0,1]^3)$ is the set of all nonempty subsets of $\tilde{\mathcal{P}}_k([0,1]^3)$, for $1 \leqslant k < n$.
3. $\tilde{\mathcal{N}}_E^{(n)} : \mathcal{X}_E \to \tilde{\mathcal{P}}_n([0,1]^3)$ is defined similarly for edge attributes $x_e \in \mathcal{X}_E$.

4. $\mathcal{R}_N = \{\mathcal{R}_1, \ldots, \mathcal{R}_K\}$ is a finite set of *neutrosophic rules*, each of the form

$$\text{IF } (T, I, F) \in S_v^{(n)} \ \wedge \ (T', I', F') \in S_u^{(n)} \ \wedge \ (T'', I'', F'') \in S_e^{(n)}$$
$$\text{THEN} \quad y_k = f_k(T, I, F, T', I', F', T'', I'', F''),$$

where $S_v^{(n)} = \tilde{\mathcal{N}}_V^{(n)}(x_v)$, $S_u^{(n)} = \tilde{\mathcal{N}}_V^{(n)}(x_u)$, and $S_e^{(n)} = \tilde{\mathcal{N}}_E^{(n)}(x_{u,v})$.

5. $\mathcal{D}_N$ is a defuzzification layer that aggregates the rule outputs into crisp predictions.

Definition 4.18 (n-SHN-GNN: Forward Pass)**.** Let $G = (V, E)$ with vertex features $X_V \subseteq \mathcal{X}_V$ and edge features $X_E \subseteq \mathcal{X}_E$. The forward computation proceeds as follows:

1. **n-SuperHyperNeutrosophic Fuzzification.**
For each $v \in V$, compute the nested set

$$S_v^{(n)} \ = \ \tilde{\mathcal{N}}_V^{(n)}(x_v) \ \subseteq \ \tilde{\mathcal{P}}_n([0,1]^3),$$

and similarly $S_e^{(n)} = \tilde{\mathcal{N}}_E^{(n)}(x_e)$ for each $e = (u, v) \in E$. By construction each bottom-level triplet (T, I, F) in these sets satisfies $0 \leqslant T + I + F \leqslant 3$.

2. **Rule Evaluation.**
For each rule $\mathcal{R}_k$ and each tuple $(v \to u)$, define its *firing strength*

$$r_k(v \to u) \ = \ \sup_{\substack{(T,I,F)\in S_v^{(n)},\\ (T',I',F')\in S_u^{(n)},\\ (T'',I'',F'')\in S_e^{(n)}}} \min\big\{\mu_{v,k}(T, I, F),\, \mu_{u,k}(T', I', F'),\, \mu_{e,k}(T'', I'', F'')\big\},$$

where each $\mu_{\bullet,k}$ is the degree to which the triplet satisfies the antecedent fuzzy pattern of $\mathcal{R}_k$.

3. **Normalization.**
Normalize the firing strengths:

$$\hat{r}_k(v \to u) \ = \ \frac{r_k(v \to u)}{\sum_{j=1}^{K} r_j(v \to u)}.$$

4. **Defuzzification and Message Passing.**
Compute the message from v to u by

$$m_{v\to u} \ = \ \sum_{k=1}^{K} \hat{r}_k(v \to u)\ f_k\big(T, I, F, T', I', F', T'', I'', F''\big),$$

and aggregate over neighbors:

$$h_u^{\text{new}} \ = \ \sigma\Big(\sum_{v\in N(u)} m_{v\to u}\Big),$$

where σ is a suitable activation function.

Example 4.19 (n-SHN-GNN: Multi-Level Sensor Reliability)**.** Consider a network of sensors $G = (V, E)$ measuring multiple indicators across sites. For each sensor v, we record uncertain reliability at three levels:

- *Level 1*: instantaneous reading uncertainty (T, I, F).
- *Level 2*: short-term consistency across readings, a set of level-1 triplets.
- *Level 3*: facility-wide aggregation, a set of level-2 sets.

Thus for $n = 3$,

$$\tilde{\mathcal{N}}_V^{(3)}(x_v) \subseteq \tilde{\mathcal{P}}_3([0,1]^3),$$

and similarly for each edge. A typical rule might be:

$$\text{IF } (T,I,F) \in S_v^{(3)} \text{ at level 2} \quad \text{AND} \quad (T',I',F') \in S_u^{(3)} \text{ at level 1} \quad \text{THEN } y_k = f_k(\dots).$$

After normalization and defuzzification, the network outputs a crisp anomaly score for each sensor, integrating multi-level uncertainties in a principled neutrosophic manner.

Theorem 4.20 (n-SHN-GNN Reduces to HN-GNN and Generalizes n-SHF-GNN)**.** *Let $n \geqslant 1$. Then:*

1. *If $n = 1$, then n-SuperHyperNeutrosophic GNN (n-SHN-GNN) of Definitions 4.17 and 4.18* coincides *with the HyperNeutrosophic GNN (HN-GNN) of Definition 4.1 (specialized to neutrosophic triplets). In symbols,*
$$\text{1-SHN-GNN} \equiv \text{HN-GNN}.$$
2. *Moreover, for arbitrary n, the n-SHN-GNN* strictly generalizes *the corresponding n-SuperHyperFuzzy GNN (n-SHF-GNN) of Definition 4.13. Concretely, by embedding every hyperfuzzy membership vector $\alpha \in [0,1]^M$ into the neutrosophic triplet*
$$(\, T = \|\alpha\|\,,\ I = 0,\ F = 0\,) \in [0,1]^3,$$
and lifting this to each nested set, one recovers exactly the n-SHF-GNN operations as the special case in which all indeterminacy and falsity components vanish.

Proof. **(1) Reduction to HN-GNN when $n = 1$.** By definition,

$$\tilde{\mathcal{P}}_1([0,1]^3) = \big\{\, A \subseteq [0,1]^3 : A \neq \varnothing \big\} = \tilde{P}([0,1]^3).$$

Hence each fuzzification $\tilde{\mathcal{N}}_V^{(1)}(x_v)$ and $\tilde{\mathcal{N}}_E^{(1)}(x_e)$ takes values in $\tilde{P}([0,1]^3)$, exactly as in the HN-GNN. The rule-evaluation step in Definition 4.18 becomes a single-level sup–min over $[0,1]^3$, matching the HN-GNN's firing-strength computation. The normalization of r_k, the defuzzification into crisp messages, and the neighbor aggregation $h_u^{\text{new}} = \sigma(\sum_v m_{v\to u})$ are carried out identically. Therefore the 1-SHN-GNN is literally the HN-GNN.

(2) Generalization of n-SHF-GNN. Recall an n-SHF-GNN uses fuzzification $\tilde{\mathcal{F}}_V^{(n)}(x) \subseteq \tilde{\mathcal{P}}_n([0,1]^M)$ and similarly for edges, with rule-firing via sup–min in $[0,1]^M$. We embed each point $\alpha = (\alpha_1, \dots, \alpha_M) \in [0,1]^M$ into the neutrosophic triplet

$$(T,I,F) = \big(\max_i \alpha_i,\ 0,\ 0\big) \in [0,1]^3,$$

so that the "truth" coordinate records the original fuzzy degree, and indeterminacy/falsity are set to zero. Extending this map to nested sets gives an injection

$$\Phi : \tilde{\mathcal{P}}_n([0,1]^M) \hookrightarrow \tilde{\mathcal{P}}_n([0,1]^3).$$

Now define the n-SHN-GNN fuzzifiers by

$$\tilde{\mathcal{N}}_V^{(n)}(x) = \Phi\big(\tilde{\mathcal{F}}_V^{(n)}(x)\big), \quad \tilde{\mathcal{N}}_E^{(n)}(x) = \Phi\big(\tilde{\mathcal{F}}_E^{(n)}(x)\big).$$

Because $I = F = 0$ everywhere, every sup–min over the neutrosophic triples reduces exactly to the sup–min over the single "truth" coordinate, which in turn agrees with the hyperfuzzy sup–min in $[0,1]^M$. The normalization and message-passing formulas likewise coincide. Thus the n-SHF-GNN is obtained by restricting the n-SHN-GNN to the special case of zero indeterminacy and falsity, proving that n-SHN-GNN strictly generalizes n-SHF-GNN. □

4.6 n-SuperHyperPlithogenic Graph Neural Network (n-SHP-GNN)

An n-SuperHyperPlithogenic Graph Neural Network (n-SHP-GNN) models uncertainty using n-level nested plithogenic membership vectors and contradiction functions for vertices and edges.

Definition 4.21 (n-SuperHyperPlithogenic GNN)**.** Let $n \geqslant 1$. An *n-SuperHyperPlithogenic Graph Neural Network* on a graph $G = (V, E)$ is a 5-tuple

$$\text{n-SHP-GNN} \;=\; \big(G,\, \tilde{\mathcal{P}}_V^{(n)},\, \tilde{\mathcal{P}}_E^{(n)},\, \mathcal{R}_P,\, \mathcal{D}_P\big),$$

where:

1. $\tilde{\mathcal{P}}_V^{(n)} : \mathcal{X}_V \to \tilde{\mathcal{P}}_n\big([0,1]^s\big)$ is the *n-SuperHyperPlithogenic vertex fuzzifier*: for each vertex attribute x_v,
$$\tilde{\mathcal{P}}_V^{(n)}(x_v) \;\subseteq\; \tilde{\mathcal{P}}_n\big([0,1]^s\big), \quad \tilde{\mathcal{P}}_V^{(n)}(x_v) \neq \varnothing.$$
Concretely, $\tilde{\mathcal{P}}_n([0,1]^s)$ is the family of nonempty nested collections of depth n built from $[0,1]^s$.
2. $\tilde{\mathcal{P}}_E^{(n)} : \mathcal{X}_E \to \tilde{\mathcal{P}}_n\big([0,1]^s\big)$ is the edge fuzzifier, defined analogously.
3. $\mathcal{R}_P$ is a finite collection of *plithogenic inference rules* combining:
   - one or more vertex membership vectors $\alpha_v \in \tilde{\mathcal{P}}_V^{(n)}(x_v)$,
   - one or more neighbor membership vectors $\alpha_u \in \tilde{\mathcal{P}}_V^{(n)}(x_u)$,
   - a *Degree of Contradiction Function* $pCF : [0,1]^s \times [0,1]^s \to [0,1]^t$.

   Each rule R_k yields a *firing strength* $\rho_k \in [0,1]$ and a *consequent output* $f_k(\alpha_v, \alpha_u, \delta) \in \mathrm{R}^d$.
4. $\mathcal{D}_P$ is a defuzzification and aggregation layer that produces a final crisp embedding or prediction.

Definition 4.22 (Forward Pass of n-SHP-GNN)**.** Given $G = (V, E)$, vertex attributes $x_v \in \mathcal{X}_V$, and edge attributes $x_{v,u} \in \mathcal{X}_E$, the network proceeds as follows:

1. **n-SuperHyperPlithogenic Fuzzification.**
   For each vertex v, compute the nested membership set
   $$A_v \;=\; \tilde{\mathcal{P}}_V^{(n)}(x_v) \;\subseteq\; \tilde{\mathcal{P}}_n([0,1]^s),$$
   and for each edge (v, u), compute
   $$B_{v,u} \;=\; \tilde{\mathcal{P}}_E^{(n)}(x_{v,u}) \;\subseteq\; \tilde{\mathcal{P}}_n([0,1]^s).$$
   By construction, each bottom-level vector $\alpha \in [0,1]^s$ satisfies $0 \leqslant \alpha_i \leqslant 1$.
2. **Rule Evaluation.**
   For each directed edge $(u \to v)$ and each inference rule R_k, define
   $$r_k(u \to v) \;=\; \sup\Big\{\min\big(\alpha_v,\; \alpha_u,\; 1 - pCF(\alpha_v, \alpha_u)\big) \;\Big|\; \alpha_v \in A_v,\; \alpha_u \in A_u\Big\},$$
   where the minimum is taken coordinate-wise in $[0,1]^s$, and $pCF(\alpha_v, \alpha_u) \in [0,1]^t$ is lifted to $[0,1]^s$ by dropping extra coordinates if necessary. The rule's *output vector* is $m_k(u \to v) = f_k(\alpha_v, \alpha_u, pCF(\alpha_v, \alpha_u))$ evaluated at the arguments achieving the supremum.
3. **Normalization.**
   For each edge $(u \to v)$, normalize the rule strengths:
   $$\hat{r}_k(u \to v) \;=\; \frac{r_k(u \to v)}{\sum_{j=1}^{K} r_j(u \to v)} \quad \text{(if the denominator} > 0, \text{ else } 0).$$

4. **Aggregation and Update.**
   Compute the aggregated message to v:

$$m_{u\to v} = \sum_{k=1}^{K} \hat{r}_k(u \to v)\, m_k(u \to v), \quad h_v \;=\; \sigma\Big(\sum_{u\in N(v)} m_{u\to v} \Big),$$

   where σ is a chosen nonlinearity (e.g. ReLU).

5. **Defuzzification / Output.**
   Finally, apply $\mathcal{D}_P$ to $\{h_v\}$ (or to the set of all $\{h_v\}$) to produce the network's prediction(s).

Example 4.23 (Multi-Level Product Bundling via n-SHP-GNN)**.** In a retail chain, let $G = (V, E)$ connect stores with strong co-purchase or promotional ties. Attributes at store v include:

$$\text{Level 1: } (\,\alpha^{(1)} \in [0,1]^s\,) \quad \text{(category demands)},$$

$$\text{Level 2: } \{\alpha_i^{(1)}\} \mapsto \alpha^{(2)} \subseteq [0,1]^s \quad \text{(sub-category aggregates)},$$

and so on up to n levels. Edge attributes $B_{v,u}$ encode historical bundling success in a similar hierarchy.

A typical plithogenic rule might read:

$$\text{IF } \alpha_v^{(1)} \in A_v, \quad \alpha_u^{(1)} \in A_u, \quad \delta = pCF(\alpha_v^{(1)}, \alpha_u^{(1)}) \quad \text{THEN} \quad m = f(\alpha_v^{(1)}, \alpha_u^{(1)}, \delta),$$

with higher-level membership sets controlling which $\alpha^{(1)}$ are considered. After normalization and aggregation, the network outputs a crisp *bundling score* $y_{v,u}$ predicting whether to offer a joint promotion at stores v and u.

This multi-nesting allows the model to distinguish, e.g., "electronics bundle well in summer" vs. "electronics fail in winter," by capturing seasonality at a higher level without re-training separate models.

Theorem 4.24 (n-SHP-GNN Reduces to HP-GNN and Generalizes n-SHN-GNN)**.** *Let $n \geqslant 1$. Then:*

(a) If $n = 1$, an n-SHP-GNN coincides exactly with an HP-GNN (Definition 4.9).

(b) For any n, every n-SHN-GNN (Definition 4.17) embeds as an n-SHP-GNN by treating each neutrosophic triplet set as a plithogenic membership set in $[0,1]^3$ with trivial contradiction function.

Proof. **(a) Reduction at $n = 1$.** By definition,

$$\tilde{\mathcal{P}}_1\big([0,1]^s\big) \;=\; \big\{\, A \subseteq [0,1]^s \mid A \neq \varnothing \big\},$$

so each fuzzifier $\tilde{\mathcal{P}}_V^{(1)}$, $\tilde{\mathcal{P}}_E^{(1)}$ takes values in exactly the same domain as the HP-GNN's hyperplithogenic fuzzifiers. Moreover:

- The rule-strength computation in Definition 4.22 becomes $\sup \min(\alpha_v, \alpha_u, 1 - pCF(\alpha_v, \alpha_u))$ over single-layer sets, which is exactly the HP-GNN's sup–min step.
- The normalization $\hat{r}_k = r_k / \sum_j r_j$ and the message aggregation $m_{u\to v} = \sum_k \hat{r}_k\, m_k$ agree term-by-term.

Hence 1-SHP-GNN $\equiv$ HP-GNN.

(b) Embedding n-SHN-GNN into n-SHP-GNN. Let

$$\text{n-SHN-GNN} \;=\; \big(G,\, \tilde{\mathcal{N}}_V^{(n)},\, \tilde{\mathcal{N}}_E^{(n)},\, \mathcal{R}_N,\, \mathcal{D}_N\big),$$

where $\tilde{\mathcal{N}}_V^{(n)}(x_v) \subseteq \tilde{\mathcal{P}}_n([0,1]^3)$ is a nested family of neutrosophic triplets (T, I, F). We construct an n-SHP-GNN

$$\text{n-SHP-GNN} = \big(G,\ \tilde{\mathcal{P}}_V^{(n)},\ \tilde{\mathcal{P}}_E^{(n)},\ \mathcal{R}_P,\ \mathcal{D}_P\big)$$

by setting:

$$\tilde{\mathcal{P}}_V^{(n)}(x) \ = \ \big\{\underbrace{(T, I, F)}_{\in [0,1]^3} \ \big|\ (T, I, F) \in \tilde{\mathcal{N}}_V^{(n)}(x)\big\},$$

and similarly for $\tilde{\mathcal{P}}_E^{(n)}$. We take the plithogenic contradiction function

$$pCF\big((T_1, I_1, F_1),\, (T_2, I_2, F_2)\big) := (\ |T_1 - T_2|,\ |I_1 - I_2|,\ |F_1 - F_2|\ ) \in [0,1]^3,$$

or any symmetric map that vanishes when the triplets coincide. Then:

- The sup–min rule-firing $\sup\min(\alpha_v, \alpha_u, 1 - pCF(\alpha_v, \alpha_u))$ over $\alpha_v, \alpha_u \in [0,1]^3$ recovers exactly the hyperneutrosophic rule-strength $\sup\min(T_v, T_u)$, $\sup\min(I_v, I_u)$, $\sup\min(F_v, F_u)$ used by the n-SHN-GNN (up to coordinating splitting).
- Normalization and message aggregation coincide, since they depend only on the numerical strengths r_k and outputs f_k.

Thus every n-SHN-GNN is realized as an n-SHP-GNN with these choices of $\tilde{\mathcal{P}}_V^{(n)}, \tilde{\mathcal{P}}_E^{(n)}, pCF$. □

5 Conclusion

In this paper, we investigated SuperHyperFuzzy Graph Neural Networks, SuperHyperNeutrosophic Graph Neural Networks, and SuperHyperPlithogenic Graph Neural Networks. While the primary focus was theoretical, we anticipate that future studies will further explore the practical implementations and real-world applications of these advanced models. In addition, we hope that further extensions employing HyperGraphs [76, 77] and SuperHyperGraphs [78–80] will be investigated in future research.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Consent to Participate

Not applicable.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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Chapter 10

TOPOLOGY INDUCED BY FUZZY COLORING IN FUZZY GRAPHS

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Abstract

In fuzzy graphs, the concept of fuzzy topology induced by fuzzy coloring on the vertex set of a fuzzy graph is a relatively advanced and specialized topic that combines the principles of fuzziness, graph theory, and topology. Fuzzy topology typically refers to the study of topological structures where the open sets are not crisp but fuzzy in nature, meaning they have degrees of membership rather than being either fully included or excluded. When fuzzy coloring is applied to set of vertices of a fuzzy graph, it leads to the creation of fuzzy topological spaces where the relationships between the vertices are characterized by fuzzy sets or fuzzy relations. This interplay between fuzzy coloring and fuzzy topology can help model systems with imprecise, uncertain, or gradated connectivity. This chapter provides the technique of inducing fuzzy topology on fuzzy graph's set of vertices induced by vertex coloring. It initiates color lower fuzzy approximation and color upper fuzzy approximation of fuzzy subgraphs and defines the closed and open sets of the fuzzy topology generated by fuzzy chromatic partition. It explores key properties of color lower fuzzy approximation and color upper fuzzy approximation of fuzzy subgraphs. It also establishes some real-time applications of the color lower fuzzy approximation and color upper fuzzy approximation.

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1 Introduction

Fuzzy graphs are mathematical structures used to model uncertainty, vagueness, and imprecision in relationships between objects or nodes. They extend the concept of traditional graphs by allowing the relationships between nodes to be characterized with varying degrees of truth or membership.Vertex coloring of fuzzy graphs is an extension of the classical concept of vertex coloring in graph theory, applied to fuzzy graphs. In vertex coloring, the goal is to assign colors to the vertices of a graph in such a way that no two adjacent vertices share the same color. In fuzzy graphs, however, the edges between vertices have fuzzy membership values, meaning the strength of the connection between vertices can vary. The concepts related to vertex coloring in fuzzy graphs are adapted to account for these varying degrees of membership. The fuzzy topology induced by fuzzy coloring on the vertex set of fuzzy graphs is a framework that uses fuzzy set theory and fuzzy relations to model the topological structure of fuzzy graphs. By combining fuzzy coloring and fuzzy topology, this approach allows for the representation of uncertain or imprecise relationships between vertices. This allows for richer modeling in areas such as decision-making, network analysis, and systems theory where precise connections and relationships are not always clear-cut. The concept of fuzzy topology induced by fuzzy coloring on the vertex set of fuzzy graphs is an emerging and complex area that blends elements of graph theory, fuzzy set theory, and topology. While there may not be an extensive set of formal theorems specifically dedicated to this intersection in existing literature, there are several relevant theorems and results related to fuzzy topological spaces, fuzzy graph coloring, and their interconnections that can be adapted or extended to the domain of fuzzy topology induced by fuzzy coloring on the vertex set of fuzzy graphs.

Allam et al. [1] looked into topological structures generated by relations those are binary and preorder and gained a topology that is quasi-discrete by symmetric relation. Salama [2] employed dominance relations to generate topological structures as well as "lower and upper approximations" and also brought into play the approach of " closure and interior operator" to induce topologies. Numerous researchers [3, 4, 5, 6] have employed different relations to generalize several topologies. In 1983 and 1986, S.Diesto, Gervacio and Guerrero [7, 8] exposed ways and means of generating topology via graphs. Later on, in 1998, Lemence and Canoy [9] exhibited the technique of inducing topology using relations on vertex set of graphs. Canoy et al. [10, 11] presented a method for generating topologies on the vertex set of graphs resulting from graph operations such as complement, join, corona, composition, and cartesian product. Recently, the authors [12] illustrated the way of producing topologies by make use of graph relations "adjacency, incidence, non-adjacency and non-incidence" and illuminated the nature of interior, closure of vertex induced subgraphs. With regard to developments, they purported the technique of establishing topologies taking relations: in-valence, out-valence, reachability in digraph in [13]. In recent developments [14], they investigated the techniques for creating topologies in graphs by utilizing graph metrics such as "distance; detour distance; circular distance and circular detour distance," and they connected the topologies produced by various metrics for specific types of graphs. In their latest work [15], they analyzed the topological frameworks created by graph grills and assessed which graph grill most closely aligns with the graph adjacency topological space. In another study, the presenters [16] exhibited a method for producing topology through graph ideals; examined the compatibility of adjacency topology with graph ideals. In contemporary research, the authors [17] developed a technique for creating topological structures through the color partition and investigated the characterization of certain new subgraphs in relation to graph chromatic partitioning. The study of fuzzy topologies induced by fuzzy coloring on fuzzy graphs is an emerging field that explores how the structural properties of graphs can be analyzed using the lens of fuzzy set theory. In this chapter, we examine the concept of fuzzy coloring on the fuzzy graphs; investigate the resulting fuzzy topologies, and explore their properties. We aim to provide a deeper understanding of how these induced topologies can be used to model and analyze complex systems, offering new perspectives in both mathematical theory and its applications in artificial intelligence.

2 Preliminaries

This section outlines the key theoretical framework, assumptions, and the definitions that guide this study. For the basic definitions of graph theory, we follow the source [18]. The following are the basic definitions related to fuzzy sets, fuzzy graphs, and fuzzy topology.

Definition 2.1. A fuzzy set is a generalization of a classical set where each element has a degree of membership, represented by a value between 0 and 1, rather than simply belonging or not belonging to the set. This degree reflects the "fuzziness" or uncertainty of membership.

Definition 2.2. A fuzzy graph is a graph whose edges assigned a fuzzy value representing the strength, intensity, or certainty of the connection between two nodes. A fuzzy graph can be denoted as $G = (V, E, \mu)$, where:
V is the set of vertices (nodes),
E is the set of edges (connections between nodes),
μ is the fuzzy membership function that assigns a value in the interval $[0, 1]$ to each edge, indicating the degree of connection between the nodes.
In some fuzzy graphs, vertices can also have fuzzy values assigned to them, which describe their level of participation or importance.

Definition 2.3. In fuzzy graphs, two vertices are considered adjacent if there exists an edge between them with membership grade greater than or equal to a certain threshold. The adjacency relationship is not binary but is based on edge's strength connecting vertices, which introduces flexibility in vertex coloring.

Definition 2.4. A fuzzy topological space is a generalization of a classical topological space. It is defined by a pair $(X, \mathcal{F})$, where X is a non-empty set, and $\mathcal{F}$ is a collection of fuzzy sets on X that satisfy certain axioms analogous to those of a topology, but with fuzzy sets instead of classical sets. The fuzzy sets in $\mathcal{F}$ represent fuzzy open sets.

Definition 2.5. A fuzzy open set is a fuzzy set A in a fuzzy topological space X that satisfies conditions similar to the axioms of open sets in classical topology, but the membership function of A is fuzzy rather than crisp. The membership function $\mu_A(x)$ indicates the degree of membership of an element $x \in X$ in the fuzzy open set A.

Definition 2.6. A fuzzy closed set is the complement of a fuzzy open set. If A is a fuzzy open set, then its complement $X \setminus A$ is a fuzzy closed set. The membership function for a fuzzy closed set typically describes the degree to which an element does not belong to the set.

3 Fuzzy Topology Produced by Fuzzy coloring

The notion of fuzzy topology induced by fuzzy coloring on the vertex set of fuzzy graphs is an emerging and complex area that blends elements of graph theory, fuzzy set theory, and topology. While there may not be an extensive set of formal theorems specifically dedicated to this intersection in existing literature, there are several relevant theorems and results related to fuzzy topological spaces, fuzzy graph coloring, and their interconnections that can be adapted or extended to the domain of fuzzy topology induced by fuzzy coloring. Below are substantial theoretical concepts and potential results that could be part of the study of this area.

Definition 3.1. A fuzzy coloring of fuzzy graph $G = (V, E, \mu)$ is a function $c : V \to [0, 1]$ that assigns to each vertex a degree of membership (or a color intensity) in the interval $[0, 1]$.

Definition 3.2. Thresholding refers to setting a specific value below which the fuzzy membership of an edge is considered negligible or unimportant. For vertex coloring, this threshold helps decide when two vertices should be colored distinctly, considering that the membership values of their connecting edge may not be high enough to enforce a strict coloring condition. In the context of fuzzy coloring, we define a fuzzy relation η on the vertex set V, where two vertices u and v are related if and only if their fuzzy colorings are equal or sufficiently close according to some threshold $\alpha \in [0,1]$. Thus, for any two vertices $u, v \in V$, we say $u\eta v$ if $c(u) = c(v)$ or $|c(u) - c(v)| \leq \alpha$, where α is a chosen threshold (in some contexts, this threshold is set to 0, meaning exact equality, and in others, a non-zero threshold is used for approximate equality).

Lemma 3.3. Fuzzy coloring is an equivalence relation.

Proof. Reflexivity:
For any vertex $u \in V$, we have $c(u) = c(u)$ by the definition of the fuzzy coloring function. Therefore, every vertex is related to itself. Thus, the relation is reflexive.
Symmetry:
Suppose $\lambda\eta\mu$. By definition of the fuzzy relation, this means that $c(\lambda) = c(\mu)$ or $|c(\lambda) - c(\mu)| \leq \alpha$ (for some threshold α). If $c(\lambda) = c(\mu)$, then clearly $c(\mu) = c(\lambda)$, so $\mu\eta\lambda$. If $|c(\lambda) - c(\mu)| \leq \alpha$, then $|c(\mu) - c(\lambda)| \leq \alpha$, so $\mu\eta\lambda$. Thus, the relation is symmetric.
Transitivity:
Suppose $\lambda\eta\mu$ and $\mu\eta\nu$. This means: $c(\lambda) = c(\mu)$ or $|c(\lambda) - c(\mu)| \leq \alpha$ and $c(\mu) = c(\nu)$ or $|c(\mu) - c(\nu)| \leq \alpha$. If $c(\lambda) = c(\mu)$ and $c(\mu) = c(\nu)$, then $c(\lambda) = c(\nu)$, so $\lambda\eta\nu$. If $|c(\lambda) - c(\mu)| \leq \alpha$ and $|c(\mu) - c(\nu)| \leq \alpha$, then by the triangle inequality: $|c(\lambda) - c(\nu)| \leq |c(\lambda) - c(\mu)| + |c(\mu) - c(\nu)| \leq \alpha + \alpha = 2\alpha$. If we choose a threshold α' such that $\alpha' = 2\alpha$, then we conclude that $|c(\lambda) - c(\nu)| \leq \alpha'$. Thus, the relation is transitive. □

We will show that a fuzzy coloring function $c : V \to [0,1]$, induces a fuzzy topology on V.

Definition 3.4. Given the fuzzy coloring function c, we can define a family of fuzzy subsets A_α for each $\alpha \in [0,1]$ based on the fuzzy coloring. The subset A_α contains all vertices whose fuzzy coloring is at least α. Specifically, for each $v \in V$, the membership function of A_α is given by: $\mu_{A_\alpha}(v) = \begin{cases} 1, & \text{if } c(v) \geq \alpha \\ 0, & \text{if } c(v) < \alpha \end{cases}$. This defines a fuzzy subset $A_\alpha \subseteq V$, which contains vertices whose membership value is greater than or equal to α. The collection $A_\alpha, \alpha \in [0,1]$ forms a fuzzy topology τ_{cf} called chromatic fuzzy topology.

Proposition 3.5. The collection $A_\alpha, \alpha \in [0,1]$, forms a fuzzy topology, unless $c(v) = 1$ for any vertex v.

Proof. First to prove $\emptyset$ and V are fuzzy subsets.
The fuzzy subset corresponding to $\alpha = 0$ is $A_0 = V$, since $\forall v \in V$, we have $c(v) \geq 0$. Thus, $\mu_{A_0}(v) = 1 \forall v \in V$, so $A_0 = V$. The fuzzy subset corresponding to $\alpha = 1$ is $A_1 = \emptyset$, since for all $v \in V$, we have $c(v) < 1$ unless $c(v) = 1$. Thus, $\mu_{A_1}(v) = 0$ for all $v \in V$, so $A_1 = \emptyset$. Thus, $A_0 = V$ and $A_1 = \emptyset$ are both fuzzy subsets of V, satisfying the property that $\emptyset$ and V are in the fuzzy topology.
Let us look at intersection of fuzzy subsets.
Consider two fuzzy subsets A_α and A_β for some $\alpha, \beta \in [0,1]$. The intersection $A_\alpha \cap A_\beta$ is a fuzzy subset defined by the minimum of the membership functions of A_α and A_β. Specifically, for each vertex $v \in V : \mu_{A_\alpha \cap A_\beta}(v) = \min(\mu_{A_\alpha}(v), \mu_{A_\beta}(v))$. This satisfies the required property of fuzzy sets, because the minimum of two fuzzy sets is also a valid fuzzy set. Therefore, the intersection of two fuzzy subsets A_α and A_β is also a fuzzy subset.
Similarly, the union of two fuzzy subsets A_α and A_β is defined by the maximum of the membership functions of A_α and A_β. For each vertex $v \in V : \mu_{A_\alpha \cup A_\beta}(v) = \max(\mu_{A_\alpha}(v), \mu_{A_\beta}(v))$. This also satisfies the required property of fuzzy sets, because the maximum of two fuzzy sets is itself a valid fuzzy set. Therefore, the union of two fuzzy subsets

A_α and A_β is also a fuzzy subset.
We have shown that:
The collection of fuzzy subsets $A_\alpha, \alpha \in [0,1]$, satisfies the properties required for a fuzzy topology:
$\emptyset$ and V are fuzzy subsets.
The intersection of any two fuzzy subsets is a fuzzy subset.
The union of any two fuzzy subsets is a fuzzy subset.
Therefore, the fuzzy coloring function c induces a fuzzy topology on the vertex set V of the fuzzy graph. □

4 Fuzzy Approximations of Fuzzy Subsets

In the context of fuzzy topological spaces induced by fuzzy coloring of fuzzy graphs, color lower fuzzy approximations and color upper fuzzy approximations are concepts used to define the boundaries of fuzzy subsets with respect to a given fuzzy topology. These approximations are crucial for understanding the behavior of fuzzy sets in fuzzy topological spaces, particularly in the context of graph theory, where fuzzy relations and memberships play a central role. The color lower and color upper fuzzy approximations of a fuzzy subset A in a fuzzy topological space provide a way to approximate the "exact" set A in terms of membership values. These concepts are especially important in fuzzy logic and rough set theory, where the set is not precisely known, and we are working with degrees of membership rather than exact set membership.

Definition 4.1. For a given fuzzy subset $A \subseteq V$, the color lower and color upper fuzzy approximations are defined as follows:
The color lower fuzzy approximation of a fuzzy set A in a chromatic fuzzy topological space corresponds to the vertices that definitely belong to the fuzzy set. Specifically, for a fuzzy set A, its color lower fuzzy approximation, $\mathbb{L}(A)$, is defined as the fuzzy set consisting of vertices whose membership value is greater than or equal to a certain threshold. Formally, the color lower fuzzy approximation of A is defined by:
$\mathbb{L}(A) = \{v \in V : c(v) \geq \alpha\}$. This means that the color lower fuzzy approximation is formed by taking all vertices that have a sufficiently high degree of membership in A. These vertices are considered "definitely" in the set.
The color upper fuzzy approximation of a fuzzy set A corresponds to the vertices that may potentially belong to the fuzzy set. It includes all vertices whose membership value is sufficiently close to being in the fuzzy set, even if their membership value is below the threshold used for the lower approximation. Formally, the color upper fuzzy approximation of A, $\mathbb{U}(A)$, is defined by:
$\mathbb{U}(A) = \{v \in V : c(v) > 0\}$. The color upper fuzzy approximation is made by including all vertices that have a non-zero membership in the fuzzy set A, even if their membership value is not high enough to be considered as part of the color lower fuzzy approximation.

Example 4.2. Consider a fuzzy graph with vertex set $\{1,2,3,4\}$. Consider a fuzzy coloring function $c : V \to [0,1]$ that assigns the following color values for the vertices in Fig.1:
$c(1) = 0.8, \quad c(2) = 0, \quad c(3) = 0.9, \quad c(4) = 0.2.$
Let?s define a fuzzy set A with a threshold $\alpha = 0.7$.
The color lower fuzzy approximation of A would include vertices whose fuzzy coloring is at least 0.7. Thus, the vertices in the color lower fuzzy approximation are 1 and 3, because their color values are 0.8 and 0.9, respectively.
The color upper fuzzy approximation of A would include vertices whose fuzzy coloring is strictly greater than 0. This includes $1,3$ and 4 .
Thus, in this case: $\mathbb{L}(A) = \{1,3\}$ and $\mathbb{U}(A) = \{1,3,4\}$

The color lower fuzzy approximations and color upper fuzzy approximations of fuzzy sets in a fuzzy topological space induced by a fuzzy coloring of a fuzzy graph have important properties that allow us to

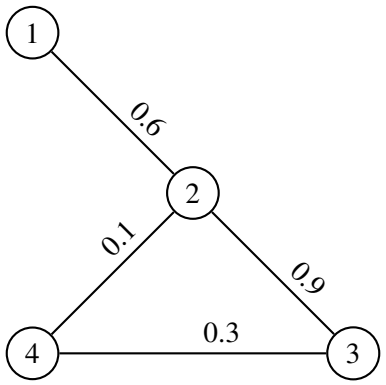

Figure 1:

understand how fuzzy subsets behave with respect to the fuzzy topology. Below are some of the key properties, along with their proofs.

Proposition 4.3. Reflexivity Property: The color lower and color upper fuzzy approximations are reflexive, meaning that the approximation of a fuzzy set is always related to itself.

Proof. The color lower and color upper approximations are based on membership values that are always at least as high as the fuzzy set itself. Let A be any fuzzy subset of V. Since every vertex v has a membership value $c(v)$, and we are considering the union over the range $\alpha \in [0,1]$, every fuzzy subset includes the vertices that have their membership values at least as high as the fuzzy set itself. Therefore, both the color lower and color upper fuzzy approximations will always contain the entire set A (or a subset thereof), ensuring reflexivity. □

Proposition 4.4. Monotonicity Property: If $A \subseteq B$, then the color lower fuzzy approximation of A is contained within the color lower fuzzy approximation of B, and similarly for the color upper fuzzy approximations.

Proof. If $A \subseteq B$, then for each vertex v, if $v \in A$, we must have $v \in B$. This means that the fuzzy coloring of each vertex in A must be at least as large as the fuzzy coloring of the corresponding vertex in B. The color lower fuzzy approximation of A corresponds to vertices with a higher or equal membership value than a threshold α, and since $A \subseteq B$, this implies that the vertices in A will also appear in the color lower fuzzy approximation of B, ensuring monotonicity for color lower fuzzy approximations. Similarly, for color upper fuzzy approximations, if $A \subseteq B$, the set of vertices whose fuzzy coloring exceeds α for A will also be included in the color upper fuzzy approximation for B. □

Proposition 4.5. Idempotency Property: The color lower and color upper approximations are idempotent, meaning that applying the approximation operator multiple times does not change the result.

Proof. For the color lower fuzzy approximation, applying it once gives the set of vertices with membership values $\mu_A(v) \geq \alpha$. Applying the color lower fuzzy approximation again will not add or remove any vertices since the membership values for vertices do not change, so the result is unchanged. Hence, the color lower fuzzy approximation is idempotent. Similarly, the color upper fuzzy approximation is idempotent. Once we apply the color upper fuzzy approximation, all the vertices that might belong to the fuzzy set A are included, and applying the operator again does not alter the set. □

Proposition 4.6. Inclusion Property: The color lower fuzzy approximation is always included in the color upper fuzzy approximation.

Proof. For any fuzzy set A, the color lower fuzzy approximation consists of the vertices whose fuzzy coloring satisfies $\mu_A(v) \geq \alpha$, meaning these vertices definitely belong to A. The color upper fuzzy approximation includes all vertices whose fuzzy coloring satisfies $\mu_A(v) > 0$, meaning these vertices are possibly in A. Since all vertices that are definitely in A (from the color lower fuzzy approximation) are also potentially in A (from the color upper fuzzy approximation), it follows that the color lower fuzzy approximation is always contained within the color upper fuzzy approximation. Color lower fuzzy approximation of $A \subseteq$ Color upper fuzzy approximation of A. □

Proposition 4.7. Convergence Property: The color lower and color upper fuzzy approximations become equal when the fuzzy set is "crisp enough" (i.e., when the set becomes more certain or the membership values converge).

Proof. As the fuzzy coloring becomes more precise (i.e., the membership values approach 0 or 1), the color lower and color upper fuzzy approximations converge. This is because when the fuzzy set is more certain (or more precise), the vertices in the set will either definitely belong to it (color lower fuzzy approximation) or not belong to it (color upper fuzzy approximation). In the case of a crisp set, where all membership values are either 0 or 1, the color lower and color upper fuzzy approximations will become equal, as every vertex is either in or out of the set with no intermediate degree of membership. □

Observation 4.8.

i. The color lower and color upper fuzzy approximations of the union and intersection of fuzzy sets behave as expected.

ii. The color lower fuzzy approximation of the union of two fuzzy sets is the union of their color lower fuzzy approximations.

iii. The color upper fuzzy approximation of the union of two fuzzy sets is the union of their color upper fuzzy approximations.

iv. The color upper fuzzy approximation of the union of two fuzzy sets is the union of their color upper fuzzy approximations.

v. The color lower fuzzy approximation of the intersection of two fuzzy sets is the intersection of their color lower fuzzy approximations.

vi. The color upper fuzzy approximation of the intersection of two fuzzy sets is the intersection of their color upper fuzzy approximations.

5 Equality of Fuzzy Approximations of Fuzzy Subsets

The color lower fuzzy approximation and color upper fuzzy approximation of fuzzy sets in fuzzy topological spaces induced by fuzzy coloring are equal when the fuzzy set becomes "crisp enough," meaning that the membership values of the vertices in the fuzzy set are either 0 or 1. This situation occurs when the fuzzy set has clear boundaries and no "fuzzy" (intermediate) membership values.

The color lower and color upper fuzzy approximations are equal when:

- The fuzzy set is crisp, meaning all vertices have membership values of either 0 or 1.

- There is no intermediate fuzzy membership (values between 0 and 1), making the fuzzy set well-defined with clear boundaries.

Why are they equal?

When the fuzzy set is crisp, the membership function for each vertex is either 0 (not in the set) or 1 (in the set). Therefore, both the color lower and color upper fuzzy approximations become identical, as there are no fractional values or ambiguities to handle:
Color Lower Approximation: Vertices whose membership value $c(v) \geq 1$ (i.e., $c(v) = 1$) will be in the set.
Color Upper Approximation: Vertices whose membership value $c(v) > 0$ will be in the set. In this case, since the membership values are either 0 or 1, both fuzzy approximations will include the same vertices, and thus they will be equal.

When are they different?

When the fuzzy set is not crisp (i.e., the membership values are in the range between 0 and 1 for some vertices), the color lower fuzzy approximation will contain fewer vertices than the color upper fuzzy approximation. The color lower fuzzy approximation will only include vertices whose membership value is at least α. The color upper fuzzy approximation will include all vertices whose membership value is strictly greater than 0.
In this case, the color upper fuzzy approximation will always be at least as large as the color lower fuzzy approximation, and they will not be equal unless the fuzzy set is crisp.

Illustrations

Illustration 1:Crisp Set (Color Lower and Color Upper Approximations are Equal) Let us consider a color-

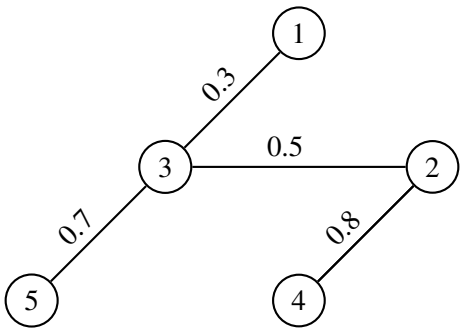

Figure 2:

ing function $c(v)$ for the vertices of the graph in Fig.2:
$c(1) = 1; c(2) = 1; c(3) = 0; c(4) = 0; c(5) = 0$.
Let A be the fuzzy set where $A = \{1,2\}$ (i.e., we want to consider the fuzzy set consisting of vertices 1 and 2, which are fully in the set with membership value 1).
For a threshold $\alpha = 0.5$, we compute: Color lower fuzzy approximation of A: Includes vertices where $c(v) \geq 1$, i.e., 1 and 2 because their membership values are 1.
$\mathbb{L}(A) = \{1,2\}$
Color Upper Fuzzy Approximation of A: Includes vertices where $c(v) > 0$, i.e., 1 and 2 because their membership values are 1.
$\mathbb{U}(A) = \{1,2\}$.

Since the fuzzy set A is crisp, both the color lower and color upper approximations are equal.

Illustration 2: Non-Crisp Set (Color Lower and Color Upper Approximations are Different).
Now, for the graph in Fig.3 consider the following fuzzy coloring function $c(v)$ for the vertices:

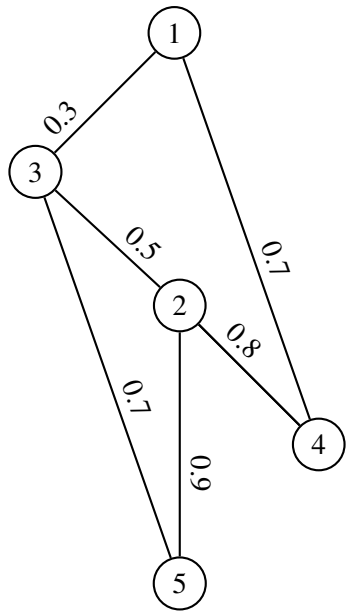

Figure 3:

$c(1) = 0.9; c(2) = 0.5; c(3) = 0.4; c(4) = 0.2; c(5) = 0.1.$
Let A be the fuzzy set where we want to consider vertices with fuzzy coloring greater than or equal to 0.5. For a threshold $\alpha = 0.5$, we compute:
Color Lower Fuzzy Approximation of A: Includes vertices where $c(v) \geq 0.5$, i.e., 1 and 2 because their membership values are 0.9 and 0.5, respectively.
$\mathbb{L}(A) = \{1,2\}$
Color Upper Fuzzy Approximation of A: Includes vertices where $c(v) > 0$, i.e., $1,2,3,4$.
$\mathbb{U}(A) = \{1,2,3,4\}$.
In this case, since the fuzzy set A is not crisp (because some membership values are not exactly 0 or 1), the color lower and color upper fuzzy approximations are different.

Illustration 3: Crisp Set with Some Vertices Having Membership 0 or 1
For the graph in Fig.4, let us consider a situation where some vertices have membership 1, and some have membership 0, but the fuzzy set is still crisp. The coloring function $c(v)$ for the vertices:
$c(1) = 1; c(2) = 0.6; c(3) = 0.7; c(4) = 0; c(5) = 0; c(6) = 0.$
Let $A = \{1\}$, a fuzzy set with only vertex 1 fully included (membership 1). For a threshold $\alpha = 0.5$, we compute:
Color Lower Fuzzy Approximation of A: Includes 1 because its membership value is 1.
$\mathbb{L}(A) = \{1,2,3\}$.
Color Upper Fuzzy Approximation of A: Includes $1,2,3$ because its membership value is greater than 0.
$\mathbb{U}(A) = \{1,2,3\}$.

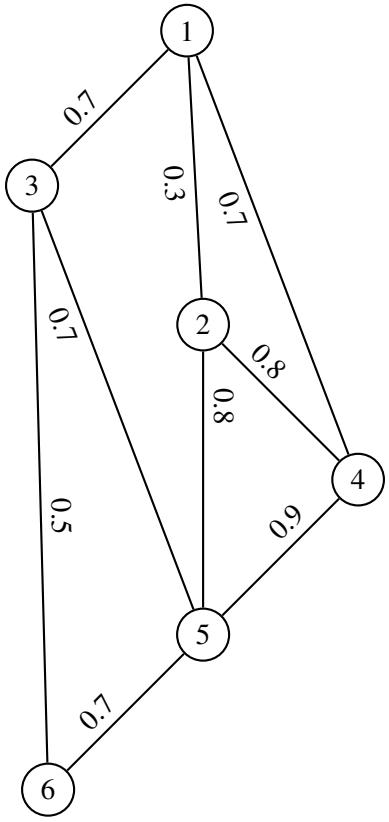

Figure 4:

In this case, even though there are other vertices with memberships between 0 and 1, the set is still crisp, so the color lower and color upper approximations are equal.

6 Applications of Fuzzy Approximations in Artificial Intelligence

The color lower fuzzy approximations and color upper fuzzy approximations of fuzzy topological spaces, induced by fuzzy coloring of fuzzy graphs, have several real-time applications in areas that involve uncertainty, imprecision, and approximation. These applications span a variety of fields such as data analysis, image processing, social networks, bioinformatics, and machine learning. The concepts of fuzzy approximations are especially useful when dealing with imprecise or incomplete information, where we need to identify boundaries or relationships between objects with uncertain membership values.

In Artificial Intelligence (AI), fuzzy logic plays a pivotal role in handling uncertainty and imprecision. The concepts of color lower fuzzy approximations and color upper fuzzy approximations in fuzzy topological spaces induced by fuzzy coloring of fuzzy graphs are particularly useful in various AI applications where knowledge is uncertain or incomplete. These approximations help AI systems make decisions, analyze data, and model relationships between entities in a way that reflects the vagueness inherent in real-world scenarios.

Key Applications of Color Lower and Color Upper Fuzzy Approximations in AI are presented in the table below: (Table 1)

Key areas where fuzzy systems integrate with AI	Nodes of the Fuzzy Graph	Color Lower Fuzzy Approximation	Color Upper Fuzzy Approximation
Medical diagnosis system	symptoms, diseases, tests, treatment options	symptoms that definitely indicate a disease	symptoms that might suggest the disease but need more evidence
Self-driving car	sensors, vehicles status, traffic signals, control systems, navigation path	decision to stop when a pedestrian is detected with certainty	decision to stop when the system detects a nearby car that is behaving unpredictably, where more data is needed to confirm the best course of action.
Image classification	image features, classifier, categories, training data, sensor data	pixels that definitely belong to particular category	pixels that are ambiguous
Robotic arm	joint Angles, motor states, force and torque, gripper state, control parameters	arm$'$s position based on precise sensor feedback	arm$'$s position might not be exact but still within an acceptable range, allowing for adaptive corrections.
Intelligent traffic management system	traffic signals, road conditions, traffic sensors	adjustment of the traffic lights to clear congestion when a high number of cars are detected at an intersection	potential adjustment when the system detects some cars waiting at a red light, but without enough certainty to make a drastic change.
Fraud detection in banking	transaction features, account behavior, historical fraud data, alerts and actions, transaction sources	transactions that definitely appear suspicious based on clear patterns	transactions that might be suspicious based on partial patterns

Table 1:

Thus the concepts of color lower fuzzy approximations and color upper fuzzy approximations of fuzzy topological spaces induced by fuzzy coloring of fuzzy graphs have numerous practical applications in AI.

7 Conclusion

In this chapter, we have explored the intricate relationship between fuzzy coloring and the fuzzy topologies created by such colorings. By introducing the concept of fuzzy coloring within the context of fuzzy graphs, we have shown how this coloring can influence the structure of the induced topological spaces, altering fundamental

properties such as openness in the graph-theoretic sense. The resulting fuzzy topologies offer a rich framework for analyzing graphs with imprecise or uncertain connections, providing a more flexible approach than classical graph theory. Further, this research paper has explored the concept of color lower and color upper fuzzy approximations of fuzzy subsets within the context of fuzzy topological spaces induced by fuzzy coloring on the vertex set of fuzzy graphs. By employing fuzzy set theory and fuzzy topology, we have developed a framework for analyzing and approximating fuzzy subsets in these spaces. Through the examination of fuzzy coloring on graphs, we have demonstrated how these color lower and color upper fuzzy approximations can offer valuable insights into the structure and behavior of fuzzy graphs, especially in applications involving uncertainty and imprecision.

Our findings suggest that the study of fuzzy topologies induced by fuzzy coloring can enhance the understanding of complex systems where relationships between components are not crisp but rather fuzzy or uncertain. This has wide-ranging implications in various fields such as network theory, fuzzy logic, and decision-making, where uncertainty plays a key role in system behavior. Furthermore, we have identified several properties of color lower and color upper fuzzy approximations that could be further investigated in future research. Despite the promising results presented in this study, there remains significant room for further exploration. Future work could delve deeper into the algebraic and topological properties of fuzzy graphs, refine the definitions of fuzzy topological structures, and investigate specific applications where these concepts can be practically applied. Additionally, extending the theory to more general types of fuzzy graphs, such as those with varying membership functions or dynamic edge relationships, could provide a broader perspective on the interactions between fuzzy coloring and fuzzy topology. In conclusion, the intersection of fuzzy topology and fuzzy graph theory opens up new possibilities for the modeling and analysis of uncertain systems, and this chapter has laid a foundation for continued exploration in this exciting and evolving area of research.

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Chapter 11

Primals Associated with Rough Sets

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ABSTRACT

This chapter explores the concept of primal approximation spaces and their foundational role in advancing rough set theory. By extending classical approximation spaces, we introduce primal structures and investigate their properties, including primal supra topology and primal approximation operators. Key results highlight the interplay between primal structures and rough approximations, offering novel insights into the topology of approximation spaces. Examples and theorems elucidate theoretical concepts, bridging the gap between abstract mathematical frameworks and potential applications. The novel approaches presented can be extended to applications in Super HyperTopology and Neutrosophic Super HyperTopology, underscoring the versatility of these methods in addressing real-world problems. Furthermore, this work bridges the gap between classical rough set models and advanced topological frameworks, providing a deeper understanding of their potential applications. These findings pave the way for future research in fields such as data analysis, decision-making, and computational modeling. Specifically, with a certain set A in an approximation space $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$, and by utilizing the primal notion, the operator $\Phi_A(B)(\boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{P}})$ is introduced. This operator generates an approximation supra-topological space.

Keywords: Primal, Rough set theory, Approximation spaces, Primal approximation spaces, Primal approximation continuity.

INTRODUCTION

Rough set theory, introduced by Pawlak [1] in 1982, provides a rigorous framework for handling uncertainty and imprecision in data analysis. It is based on approximation spaces $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$, where $\boldsymbol{W}$ is a finite set of objects and $\boldsymbol{\mathcal{R}}$ is an equivalence relation. Rough sets are described by their lower and upper approximations, which delineate the boundary of uncertainty for a set. A set is considered exact when its lower and upper approximations coincide, indicating the absence of roughness.

Rough set theory has been applied across various domains [2–5], often in combination with topological structures. Notable applications include fuzzy topological spaces [6], topological rough sets [7], bipolar hypersoft sets [8], medical analysis [9–12], and decision-making problems [13–15]. Topology continues to serve as a fundamental framework for modeling uncertainty, as emphasized by Sierpiński in [16].

The interplay between rough sets and topology has attracted significant attention. Thivagar and Richard [17] introduced nano-topological spaces defined via lower and upper approximations. Subsequent research has extended nano topology [18, 19, 20] and employed concepts such as ideals [21] and grills [22] to study advanced structures, including proximity, uniform, and closure spaces, with further applications reported in [23–27].

In 2022, Acharjee et al. [28] proposed primal topology, based on a primal structure dual to grills. Building on this concept, we introduce **primal approximation spaces**, which integrate primal structures with classical rough approximations. These structures, characterized by their algebraic and topological properties, provide a refined approach to uncertainty analysis.

This chapter presents the operator $\Phi_A(B)(\boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{P}})$, defined for a set A in an approximation space $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$, which generates an approximation supra-topological space. We establish foundational results, introduce primal approximation operators, and demonstrate how they link primal supra-topology with standard approximation spaces through theorems, corollaries, and examples.
Finally, we extend the discussion to advanced frameworks such as **Super HyperTopology** and **Neutrosophic Super HyperTopology**, highlighting their effectiveness in modeling complex problems. The proposed primal approximation spaces mark a significant development in rough set theory, opening new directions for both theoretical research and real-world applications.

BACKGROUND

In the line of this article, consider $\boldsymbol{W}$ is a finite set of items as a universal set, $\mathbf{2}^{\boldsymbol{W}}$ indicates the class of all subsets of $\boldsymbol{W}$, and $\boldsymbol{\mathcal{R}}$ is an equivalence relation on $\boldsymbol{W}$. Later, the pair $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$ is named an approximation space. The equivalence class $[x]_{\boldsymbol{\mathcal{R}}}$ of an element $x \in \boldsymbol{W}$ is assigned by $[x]_{\boldsymbol{\mathcal{R}}} = \{y \in \boldsymbol{W} : (x, y) \in \boldsymbol{\mathcal{R}}\}$. For a subset A of $\boldsymbol{W}$, the lower approximation $L_{\boldsymbol{\mathcal{R}}}(A)$, the upper approximation $U_{\boldsymbol{\mathcal{R}}}(A)$, and the boundary region $B_{\boldsymbol{\mathcal{R}}}(A)$ [1] are identified as follows:

$$L_{\boldsymbol{\mathcal{R}}}(A) = \bigcup_{x \in \boldsymbol{W}} \{[x]_{\boldsymbol{\mathcal{R}}} : [x]_{\boldsymbol{\mathcal{R}}} \subseteq A\},$$

$$U_{\boldsymbol{\mathcal{R}}}(A) = \cup_{x \in \boldsymbol{W}} \{[x]_{\boldsymbol{\mathcal{R}}} : [x]_{\boldsymbol{\mathcal{R}}} \cap A \neq \emptyset\}, \text{ and}$$

$$B_{\boldsymbol{\mathcal{R}}}(A) = U_{\boldsymbol{\mathcal{R}}}(A) \backslash L_{\boldsymbol{\mathcal{R}}}(A).$$

Lemma 1 [1] Let $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$ be an approximation space. For any sets $A, \acute{A} \subseteq \boldsymbol{W}$, then the following statements hold:

1) $L_{\boldsymbol{\mathcal{R}}}(\mathrm{A}) \subseteq \mathrm{A} \subseteq U_{\boldsymbol{\mathcal{R}}}(\mathrm{A})$.
2) $L_{\boldsymbol{\mathcal{R}}}(\emptyset) = U_{\boldsymbol{\mathcal{R}}}(\emptyset) = \emptyset$ *and* $L_{\boldsymbol{\mathcal{R}}}(\boldsymbol{W}) = U_{\boldsymbol{\mathcal{R}}}(\boldsymbol{W}) = \boldsymbol{W}$.
3) $L_{\boldsymbol{\mathcal{R}}}(\mathrm{A}) \subseteq L_{\boldsymbol{\mathcal{R}}}(\acute{A})$ *and* $U_{\boldsymbol{\mathcal{R}}}(\mathrm{A}) \subseteq U_{\boldsymbol{\mathcal{R}}}(\acute{A})$, *if* $\mathrm{A} \subseteq \acute{A}$.
4) $L_{\boldsymbol{\mathcal{R}}}(\mathrm{A} \cup \acute{A}) \supseteq L_{\boldsymbol{\mathcal{R}}}(\mathrm{A}) \cup L_{\boldsymbol{\mathcal{R}}}(\acute{A})$, *and* $U_{\boldsymbol{\mathcal{R}}}(\mathrm{A} \cap \acute{A}) \subseteq U_{\boldsymbol{\mathcal{R}}}(\mathrm{A}) \cap U_{\boldsymbol{\mathcal{R}}}(\acute{A})$.
5) $L_{\boldsymbol{\mathcal{R}}}(\mathrm{A} \cap \acute{A}) = L_{\boldsymbol{\mathcal{R}}}(A) \cap L_{\boldsymbol{\mathcal{R}}}(\acute{A})$ *and* $U_{\boldsymbol{\mathcal{R}}}(\mathrm{A} \cup \acute{A}) = U_{\boldsymbol{\mathcal{R}}}(\mathrm{A}) \cup U_{\boldsymbol{\mathcal{R}}}(\acute{A})$.
6) $L_{\boldsymbol{\mathcal{R}}}(\mathrm{A}^c) = (U_{\boldsymbol{\mathcal{R}}}(A))^c$ *and* $(L_{\boldsymbol{\mathcal{R}}}(A))^c = U_{\boldsymbol{\mathcal{R}}}(\mathrm{A}^c)$, *where* A^c *denotes the complement of* A *in* $\boldsymbol{W}$.
7) $U_{\boldsymbol{\mathcal{R}}}\big(L_{\boldsymbol{\mathcal{R}}}(\mathrm{A})\big) = L_{\boldsymbol{\mathcal{R}}}(L_{\boldsymbol{\mathcal{R}}}(\mathrm{A})) = L_{\boldsymbol{\mathcal{R}}}(\mathrm{A})$.
8) $L_{\boldsymbol{\mathcal{R}}}\big(U_{\boldsymbol{\mathcal{R}}}(\mathrm{A})\big) = U_{\boldsymbol{\mathcal{R}}}(U_{\boldsymbol{\mathcal{R}}}(\mathrm{A})) = U_{\boldsymbol{\mathcal{R}}}(\mathrm{A})$.

A nano topology $\boldsymbol{\tau_A}$ was introduced on $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$ [17] and is identified as follows:

Definition 2 [17] Let $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$ be an approximation space associated with $\mathrm{A} \subseteq \boldsymbol{W}$. A Nano topology $\boldsymbol{\tau_A}$ is defined on $\boldsymbol{W}$ as $\boldsymbol{\tau_A} = \{\emptyset, \boldsymbol{W}, L_{\boldsymbol{\mathcal{R}}}(\mathrm{A}), U_{\boldsymbol{\mathcal{R}}}(\mathrm{A}), B_{\boldsymbol{\mathcal{R}}}(\mathrm{A})\}$.

In 2023, Ibedou et. al. [29] offered $int^A_{\boldsymbol{\mathcal{R}}}$, $cl^A_{\boldsymbol{\mathcal{R}}}$ associated with a subset A of $\boldsymbol{W}$ in an approximation space $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$.

Definition 3 [29] In an approximation space $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$, the interior operator associated with $\mathrm{A} \subseteq \boldsymbol{W}$, $int^A_{\boldsymbol{\mathcal{R}}} : \mathbf{2}^{\boldsymbol{W}} \to \mathbf{2}^{\boldsymbol{W}}$, is offered as:

$$int^A_{\boldsymbol{\mathcal{R}}}(B) = \begin{cases} L_{\boldsymbol{\mathcal{R}}}(\mathrm{A}) \cap L_{\boldsymbol{\mathcal{R}}}(B), & B \neq \boldsymbol{W} \\ \boldsymbol{W}, & B = \boldsymbol{W} \end{cases}$$

Moreover, the closure operator $cl^A_{\boldsymbol{\mathcal{R}}} : \mathbf{2}^{\boldsymbol{W}} \to \mathbf{2}^{\boldsymbol{W}}$, is assigned as:

$$cl_{\mathcal{R}}^{A}(B) = \begin{cases} (L_{\mathcal{R}}(\mathrm{A}))^{c} \cup U_{\mathcal{R}}(B), & B \neq \emptyset \\ \emptyset, & B = \emptyset \end{cases}$$

Lemma 4 [29] Let $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$ be an approximation space associated with $\mathrm{A} \subseteq \boldsymbol{W}$. *If* $B, \acute{B}$ are subsets of $\boldsymbol{W}$, then the following statements are valid:

1) $int_{\mathcal{R}}^{A}(\emptyset) = cl_{\mathcal{R}}^{A}(\emptyset) = \emptyset$, $int_{\mathcal{R}}^{A}(\boldsymbol{W}) = cl_{\mathcal{R}}^{A}(\boldsymbol{W}) = \boldsymbol{W}$,

2) $int_{\mathcal{R}}^{A}(B) \subseteq B \subseteq cl_{\mathcal{R}}^{A}(B)$,

3) $int_{\mathcal{R}}^{A}(B) \subseteq int_{\mathcal{R}}^{A}(\acute{B})$, $cl_{\mathcal{R}}^{A}(B) \subseteq cl_{\mathcal{R}}^{A}(\acute{B})$, *if* $B \subseteq \acute{B}$,

4) $int_{\mathcal{R}}^{A}(B \cap \acute{B}) = int_{\mathcal{R}}^{A}(B) \cap int_{\mathcal{R}}^{A}(\acute{B})$, *and* $int_{\mathcal{R}}^{A}(B \cup \acute{B}) = int_{\mathcal{R}}^{A}(B) \cup int_{\mathcal{R}}^{A}(\acute{B})$,

5) $cl_{\mathcal{R}}^{A}(B \cap \acute{B}) = cl_{\mathcal{R}}^{A}(B) \cap cl_{\mathcal{R}}^{A}(\acute{B})$, *and* $cl_{\mathcal{R}}^{A}(B \cup \acute{B}) = cl_{\mathcal{R}}^{A}(B) \cup cl_{\mathcal{R}}^{A}(\acute{B})$,

6) $int_{\mathcal{R}}^{A}(int_{\mathcal{R}}^{A}(B)) = int_{\mathcal{R}}^{A}(B)$, *and* $cl_{\mathcal{R}}^{A}(cl_{\mathcal{R}}^{A}(B)) = cl_{\mathcal{R}}^{A}(B)$,

7) $int_{\mathcal{R}}^{A}(B^{c}) = (cl_{\mathcal{R}}^{A}(B))^{c}$, and $cl_{\mathcal{R}}^{A}(B^{c}) = (int_{\mathcal{R}}^{A}(B))^{c}$.

Thus, the operators $int_{\mathcal{R}}^{A}$, $cl_{\mathcal{R}}^{A}$ induce a topology in $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$ which is specified as:

$$\boldsymbol{T}_{\mathcal{R}}^{A} = \{B^{c} \subseteq \boldsymbol{W} \mid B = cl_{\mathcal{R}}^{A}(B)\} \quad \ldots \text{Eq. (1)}$$

Note that: No correlation exists between the Nano topology $\boldsymbol{\tau_A}$ established on the set $\boldsymbol{W}$ and the approximation topology $\boldsymbol{T}_{\mathcal{R}}^{A}$ derived from the interior operator, as outlined in Eq. (1).

Note that:

$$U_{\mathcal{R}}(cl_{\mathcal{R}}^{A}(B)) = cl_{\mathcal{R}}^{A}(U_{\mathcal{R}}(B)) = cl_{\mathcal{R}}^{A}(B), \quad L_{\mathcal{R}}(int_{\mathcal{R}}^{A}(B)) = int_{\mathcal{R}}^{A}(L_{\mathcal{R}}(B)) = int_{\mathcal{R}}^{A}(B), \forall B \subseteq \boldsymbol{W}.$$

Definition 5 [29] Let $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$ and $(\boldsymbol{\mathcal{C}}, \boldsymbol{S})$ be two approximation spaces associated with $\boldsymbol{A} \subseteq \boldsymbol{W}, \boldsymbol{E} \subseteq \boldsymbol{\mathcal{C}}$, respectively. A mapping *f:* $(\boldsymbol{W}, \boldsymbol{\mathcal{R}}) \rightarrow (\boldsymbol{\mathcal{C}}, \boldsymbol{S})$ is named:

1) approximation continuous, if $f^{-1}(int_{S}^{E}(\mathrm{U})) \subseteq int_{\mathcal{R}}^{A}(f^{-1}(\mathrm{U}))$, $\forall \mathrm{U} \subseteq \boldsymbol{\mathcal{C}}$. It is equivalent to the condition $cl_{\mathcal{R}}^{A}(f^{-1}(\mathrm{U})) \subseteq f^{-1}(cl_{S}^{E}(\mathrm{U}))$, $\forall \mathrm{U} \subseteq \boldsymbol{\mathcal{C}}$.

2) approximation open, if $f(int_{\mathcal{R}}^{A}(\mathrm{M})) \subseteq int_{S}^{E}(f(\mathrm{M}))$ $\forall \mathrm{M} \subseteq \boldsymbol{W}$.

Definition 6 [29] Associated with a set A, an approximation space $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$ is named:

1)$\boldsymbol{T_0}$**-space**, if for all $a, b \in A$, with $a \neq b$ there exists $M \subseteq \boldsymbol{W}$ such that
$$a \in int_{\mathcal{R}}^{A}(M), b \notin M.$$

2)$\boldsymbol{T_1}$**-space**, if for all $a, b \in A$, with there exist $M, H \subseteq \boldsymbol{W}$ such that
$$a \in int_{\mathcal{R}}^{A}(M), b \notin M \text{ and } b \in int_{\mathcal{R}}^{A}(H), a \notin H.$$

3) $\boldsymbol{T_2}$**-space**, if for all $a, b \in A$, with $a \neq b$ there exist $M, H \subseteq \boldsymbol{W}$ such that
$$a \in int_{\mathcal{R}}^{A}(M), b \in int_{\mathcal{R}}^{A}(H), and\ M \cap H = \emptyset.$$

Definition 7 [21] Let $\boldsymbol{W} \neq \emptyset$. A class $\boldsymbol{\mathcal{I}} \subseteq 2^{\boldsymbol{W}}$ is named an ideal on $\boldsymbol{W}$, if it satisfies the next conditions:

1) $B \in \boldsymbol{\mathcal{I}}$ *and* $\acute{B} \subseteq B \Rightarrow \acute{B} \in \boldsymbol{\mathcal{I}}$,

2) $B \in \boldsymbol{\mathcal{I}}$ *and* $\acute{B} \in \boldsymbol{\mathcal{I}} \Rightarrow B \cup \acute{B} \in \boldsymbol{\mathcal{I}}$.

Definition 8 [28] Let $\boldsymbol{W} \neq \emptyset$. A class $\boldsymbol{\mathcal{P}} \subseteq 2^{\boldsymbol{W}}$ is named a primal on $\boldsymbol{W}$, if it satisfies the next conditions:

1) $\boldsymbol{W} \notin \boldsymbol{\mathcal{P}}$;

2) $B \in \boldsymbol{\mathcal{P}}$ *and* $\acute{B} \subseteq B \Rightarrow \acute{B} \in \boldsymbol{\mathcal{P}}$,

3) $B \cap \acute{B} \in \boldsymbol{\mathcal{P}} \Leftrightarrow B \in \boldsymbol{\mathcal{P}}$ *or* $\acute{B} \in \boldsymbol{\mathcal{P}}$.

Lemma 9 [30, 31] Let $\boldsymbol{\mathcal{P}}$ be a primal on $\boldsymbol{W}$. $\boldsymbol{B \notin \mathcal{P}}$ $\boldsymbol{and}$ $\boldsymbol{\acute{B} \notin \mathcal{P} \Leftrightarrow B \cap \acute{B} \notin \mathcal{P}}$.

Remark 10 [30, 31] The set $\boldsymbol{\mathcal{P}} = \{\emptyset\}$ cannot act as a primal over any universe. For example, let $\boldsymbol{W} = \{\mathrm{a}, \mathrm{b}\}$. Here, the intersection of {a} and {b} is the empty set, which belongs to $\boldsymbol{\mathcal{P}}$, even though neither {a} nor {b} is contained in $\boldsymbol{\mathcal{P}}$.

Remark 11 [30, 31] When the universe consists of a single element, such as W ={a}, it is possible to construct an ideal, however, constructing a primal is not feasible. To construct a primal, the set W must contain at least two distinct elements.

Remark 12 [28] The family obtained by taking the intersection or union of two primals does not necessarily form a primal, as demonstrated in the next example:

Example 13 Let $\boldsymbol{\mathcal{P}}$ = {∅, {a}, {b}, {d}, {a, b}, {b, d}, {a, d}, {a, b, d}}, $\boldsymbol{Q}$ = {∅, {a}, {b}, {c},{a, b}, {b, c}, {a, c},{a, b, c}} be two primals on $\boldsymbol{W}$ ={a, b, c, d}. Then,

1) $\boldsymbol{\mathcal{P}} \cup \boldsymbol{Q} = \mathbf{2}^{\boldsymbol{W}} \setminus \{\boldsymbol{W}, \{a, c, d\}, \{b, c, d\}, \{c, d\}\}$ is primal.

2) The family $\boldsymbol{\Omega} = \{\boldsymbol{M} \cap \boldsymbol{H} : \boldsymbol{M} \in \boldsymbol{\mathcal{P}}, \boldsymbol{H} \in \boldsymbol{Q}\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ is not a primal on $\boldsymbol{W}$, since {a, b, c} ∩ {a, b, d} = {a, b} ∈ $\boldsymbol{\nabla}$ but neither {a, b, c} ∈ $\boldsymbol{\Omega}$ nor {a, b, d} ∈ $\boldsymbol{\Omega}$.

3) The family $\mho = \{\boldsymbol{M} \cup \boldsymbol{H} : \boldsymbol{M} \in \boldsymbol{\mathcal{P}}, \boldsymbol{H} \in \boldsymbol{Q}\} = \mathbf{2}^{\boldsymbol{W}}$ is not a primal on $\boldsymbol{W}$, since $\boldsymbol{W} \in \mho$.

Lemma 14 Let $\boldsymbol{W} \neq \emptyset$. Then the following families are primals on $\boldsymbol{W}$:

1) [28] $\mathbf{2}^{\boldsymbol{W}} \setminus \{\boldsymbol{W}\}$ (trivial primal).

2) [30, 31] $\boldsymbol{\mathcal{P}}_{\{y\}} = \{\mathbf{M} \subseteq \mathbf{W} : \mathbf{y} \notin \mathbf{M}\}$ (excluded point primal).

3) [30, 31] $\boldsymbol{\mathcal{P}}_{\boldsymbol{O}} = \{\mathbf{M} \subseteq \mathbf{W} : \mathbf{M} \cup O \neq \mathbf{W}\}$ (modified excluded set primal).

Definition 15 [30, 31] Let $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$ be an approximation space and $\boldsymbol{\mathcal{P}}$ be a primal on $\boldsymbol{W}$. Then, the triple $(\boldsymbol{W}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{P}})$ is called a primal approximation space.

PRIMAL APPROXIMATION SPACES

Definition 16 Let $(\boldsymbol{W}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{P}})$ be a primal approximation space associated with $A \subseteq \boldsymbol{W}$. Then, the local set $\Phi_A(B)(\boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{P}})$ of a set $B \subseteq \boldsymbol{W}$ with respect to A is identified as:

$$\Phi_A(B)(\boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{P}}) = \bigcap \{G \subseteq \boldsymbol{W} \mid (B \backslash G) \in \boldsymbol{\mathcal{P}}, cl^A_{\boldsymbol{\mathcal{R}}}(G) = G\}.$$

For brevity, $\Phi^A_{\boldsymbol{\mathcal{P}}}(B)$ will be used in place of $\Phi_A(B)(\boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{P}})$.

Proposition 17 Let $(\boldsymbol{W}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{P}})$ be a primal approximation space associated with $A \subseteq \boldsymbol{W}$. For any subsets B, C of $\boldsymbol{W}$:

1) $\Phi^A_{\boldsymbol{\mathcal{P}}}(B) \subseteq B$, *if* $B = cl^A_{\boldsymbol{\mathcal{R}}}(B)$.

2) $B \cap \Phi^A_{\boldsymbol{\mathcal{P}}}(C) \subseteq \Phi^A_{\boldsymbol{\mathcal{P}}}(B \cap C)$, if $B = int^A_{\boldsymbol{\mathcal{R}}}(B)$.

Proof:

1) Straightforward.
2) Suppose that $q \notin \Phi^A_{\boldsymbol{\mathcal{P}}}(B \cap C) = \bigcap \{G \subseteq \boldsymbol{W} \mid ((B \cap C) \backslash G) \in \boldsymbol{\mathcal{P}}, cl^A_{\boldsymbol{\mathcal{R}}}(G) = G\}$. Hence there exists $G \subseteq \boldsymbol{W}$ with $q \notin G$, $(B \cap C) \backslash G \in \boldsymbol{\mathcal{P}}$, and $cl^A_{\boldsymbol{\mathcal{R}}}(G) = G$. Since $B^c = cl^A_{\boldsymbol{\mathcal{R}}}(B^c)$, then $(C \backslash (B^c \cup G) \in \boldsymbol{\mathcal{P}}$, and $cl^A_{\boldsymbol{\mathcal{R}}}(B^c \cup G) = B^c \cup G$ for some $q \notin (B^c \cup G)$. Consequently, $q \notin B \cap \Phi^A_{\boldsymbol{\mathcal{P}}}(C)$ and so $B \cap \Phi^A_{\boldsymbol{\mathcal{P}}}(C) \subseteq \Phi^A_{\boldsymbol{\mathcal{P}}}(B \cap C)$.

Remark 18 Let $(\boldsymbol{W}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{P}})$ be a primal approximation space associated with $A \subseteq \boldsymbol{W}$. For any subset B of $\boldsymbol{W}$, $\Phi^A_{\boldsymbol{\mathcal{P}}}(B) \subseteq B$ or $\Phi^A_{\boldsymbol{\mathcal{P}}}(B) \supseteq B$ do not be true as displayed by the next example.

Example 19 Let $\boldsymbol{W} = \{a, b, c\}$, and $\boldsymbol{W} \backslash \boldsymbol{\mathcal{R}} = \{\{a\}, \{b, c\}\}$. If $A = \{a, b\}$, then $L_{\boldsymbol{\mathcal{R}}}(A) = \{a\}$, $(L_{\boldsymbol{\mathcal{R}}}(A))^c = \{b, c\}$. Hence, $\boldsymbol{\tau_A} = \{\emptyset, \boldsymbol{W}, \{a\}, \{b, c\}\}$ and $\boldsymbol{T^A_{\mathcal{R}}} = \{\emptyset, \boldsymbol{W}, \{a\}\}$. Consider $\boldsymbol{\mathcal{P}} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ is a primal on $\boldsymbol{W}$. Thus, from Table 1:

- If $B = \{a\}$, then $\Phi^A_{\boldsymbol{\mathcal{P}}}(B) = \emptyset$ and so $B \not\subseteq \Phi^A_{\boldsymbol{\mathcal{P}}}(B)$;
- If $B = \{b\}$, then $\Phi^A_{\boldsymbol{\mathcal{P}}}(B) = \{b, c\}$ and so $\Phi^A_{\boldsymbol{\mathcal{P}}}(B) \not\subseteq B$.

Table 1: Key operators applied in Example 19.

$B \subseteq \boldsymbol{W}$	$L_{\mathcal{R}}(B)$	$U_{\mathcal{R}}(B)$	$int^{A}_{\mathcal{R}}(B)$	$cl^{A}_{\mathcal{R}}(B)$	$\Phi^{A}_{\mathcal{P}}(B)$	$cl^{A}_{\Phi_{\mathcal{P}}}(B)$	$\Phi^{A}_{\mathcal{Q}}(B)$	$cl^{A}_{\Phi_{\mathcal{Q}}}(B)$
$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\boldsymbol{W}$	$\emptyset$	$\{a\}$	$\emptyset$	$\{a\}$
$\{b\}$	$\emptyset$	$\{b,c\}$	$\emptyset$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	$\emptyset$	$\{b\}$
$\{c\}$	$\emptyset$	$\{b,c\}$	$\emptyset$	$\{b,c\}$	$\emptyset$	$\{c\}$	$\emptyset$	$\{c\}$
$\{a,b\}$	$\{a\}$	$\boldsymbol{W}$	$\{a\}$	$\boldsymbol{W}$	$\{b,c\}$	$\boldsymbol{W}$	$\emptyset$	$\{a,b\}$
$\{a,c\}$	$\{a\}$	$\boldsymbol{W}$	$\{a\}$	$\boldsymbol{W}$	$\emptyset$	$\{a,c\}$	$\emptyset$	$\{a,c\}$
$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	$\emptyset$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\boldsymbol{W}$	$\boldsymbol{W}$	$\boldsymbol{W}$	$\boldsymbol{W}$	$\boldsymbol{W}$	$\{b,c\}$	$\boldsymbol{W}$	$\{b,c\}$	$\boldsymbol{W}$

Theorem 20 Let $(\boldsymbol{W},\mathcal{R},\mathcal{P})$ be a primal approximation space associated with $A \subseteq \boldsymbol{W}$. If $B, \acute{B} \subseteq \boldsymbol{W}$, *then*

1) $\Phi^{A}_{\mathcal{P}}(\emptyset) = \emptyset$.
2) $B \subseteq \acute{B} \Rightarrow \Phi^{A}_{\mathcal{P}}(B) \subseteq \Phi^{A}_{\mathcal{P}}(\acute{B})$.
3) $\Phi^{A}_{\mathcal{P}}(B) \cup \Phi^{A}_{\mathcal{P}}(\acute{B}) \subseteq \Phi^{A}_{\mathcal{P}}(B \cup \acute{B})$.
4) $\Phi^{A}_{\mathcal{P}}\left(B \cap \acute{B}\right) = \Phi^{A}_{\mathcal{P}}(B) \cap \Phi^{A}_{\mathcal{P}}(\acute{B})$.
5) $\Phi^{A}_{\mathcal{P}}(B) \subseteq cl^{A}_{\mathcal{R}}(B)$.
6) $int^{A}_{\mathcal{R}}\left(\Phi^{A}_{\mathcal{P}}(B)\right) \subseteq \Phi^{A}_{\mathcal{P}}(B) = cl^{A}_{\mathcal{R}}\left(\Phi^{A}_{\mathcal{P}}(B)\right)$. Moreover, $\Phi^{A}_{\mathcal{P}}$(B) is $\mathrm{T}^{\mathrm{A}}_{\mathcal{R}}$ closed.
7) $\Phi^{A}_{\mathcal{P}}(\Phi^{A}_{\mathcal{P}}(B)) \subseteq \Phi^{A}_{\mathcal{P}}(B)$.

Proof:

1) Since $cl^{A}_{\mathcal{R}}(\emptyset) = \emptyset$, then $\Phi^{A}_{\mathcal{P}}(\emptyset) = \bigcap\{G \subseteq \boldsymbol{W} \mid (\emptyset \backslash G) \in \mathcal{P},\ cl^{A}_{\mathcal{R}}(G) = G\} = \emptyset$.
2) Suppose that $\Phi^{A}_{\mathcal{P}}(B) \nsubseteq \Phi^{A}_{\mathcal{P}}(\acute{B})$. Then, there exists $G \subseteq \boldsymbol{W}$ with $\acute{B} \backslash G \in \mathcal{P}$ and $cl^{A}_{\mathcal{R}}(G) = G$ s.t. $\Phi^{A}_{\mathcal{P}}(B) \nsubseteq G$. Since $B \subseteq \acute{B}$, then $B \backslash G \subseteq \acute{B} \backslash G$. So $B \backslash G \in \mathcal{P}$, $cl^{A}_{\mathcal{R}}(G) = G$. Thus, $\Phi^{A}_{\mathcal{P}}(B) \subseteq G$, which is a contradiction and hence $\Phi^{A}_{\mathcal{P}}(B) \subseteq \Phi^{A}_{\mathcal{P}}(\acute{B})$.
3) According to (2), $\Phi^{A}_{\mathcal{P}}(B) \cup \Phi^{A}_{\mathcal{P}}(\acute{B}) \subseteq \Phi^{A}_{\mathcal{P}}(B \cup \acute{B})$.
4) Obvious.
5) Since $B \backslash cl^{A}_{\mathcal{R}}(B) = \emptyset$ and $cl^{A}_{\mathcal{R}}(cl^{A}_{\mathcal{R}}(B)) = cl^{A}_{\mathcal{R}}(B)$, then $\Phi^{A}_{\mathcal{P}}(B) \subseteq cl^{A}_{\mathcal{R}}(B)$, for any primal $\mathcal{P}$.
6) Obvious.
7) Since $\Phi^{A}_{\mathcal{P}}$(B) is $\mathrm{T}^{\mathrm{A}}_{\mathcal{R}}$ closed, it follows that $\Phi^{A}_{\mathcal{P}}(B) = cl^{A}_{\mathcal{R}}\left(\Phi^{A}_{\mathcal{P}}(B)\right)$. According to item (1) of Proposition 17, we then have $\Phi^{A}_{\mathcal{P}}(\Phi^{A}_{\mathcal{P}}(B)) \subseteq \Phi^{A}_{\mathcal{P}}(B)$.

The inclusion given in (3) of Theorem 20 need not be reversible as shown by next example:

Example 21 Refer to Example 19 for continuation. Let $\boldsymbol{Q} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\}$ be a primal on $\boldsymbol{W}$. If $B = \{b\}, \acute{B} = \{c\}$, then according to Table 1: $\Phi^{A}_{\mathcal{Q}}\left(B \cup \acute{B}\right) = \{b,c\}, \Phi^{A}_{\mathcal{Q}}(B) = \emptyset, \Phi^{A}_{\mathcal{Q}}\left(\acute{B}\right) = \emptyset$, and so $\Phi^{A}_{\mathcal{Q}}(B) \cup \Phi^{A}_{\mathcal{Q}}\left(\acute{B}\right) = \emptyset$. Consequently, $\Phi^{A}_{\mathcal{Q}}\left(B \cup \acute{B}\right) \nsubseteq \Phi^{A}_{\mathcal{Q}}(B) \cup \Phi^{A}_{\mathcal{Q}}\left(\acute{B}\right)$.

The items outlined in the following Remark will be addressed as indicated in Table 1.

Remark 22 Let $(\boldsymbol{W},\mathcal{R},\mathcal{P})$ be a primal approximation space associated with $A \subseteq \boldsymbol{W}$. If $B \subseteq \boldsymbol{W}$, then

1) $\Phi_{\mathcal{P}}^{A}(\boldsymbol{W}) \neq \boldsymbol{W}$;

2) $cl_{\mathcal{R}}^{A}(B) \nsubseteq \Phi_{\mathcal{P}}^{A}(B)$; and

3) $\Phi_{\mathcal{P}}^{A}(B) \nsubseteq int_{\mathcal{R}}^{A}(\Phi_{\mathcal{P}}^{A}(B))$.

Remark 23 Let $(\boldsymbol{W}, \mathcal{R}, \mathcal{P})$ be a primal approximation space associated with $A \subseteq \boldsymbol{W}$. For any subset B of $\boldsymbol{W}$, $\Phi_{\mathcal{P}}^{A}(B) \subseteq \Phi_{\mathcal{P}}^{A}(\Phi_{\mathcal{P}}^{A}(B))$, do not be true as displayed by the next example:

Example 24 Consider $\mathcal{P} = 2^{\boldsymbol{W}} \backslash \{\boldsymbol{W}, \{b,d,e\}, \{a,b,d,e\}, \{b,c,d,e\}\}$ is a primal on $\boldsymbol{W} = \{a,b,c,d,e\}$, and $\boldsymbol{W} \backslash \mathcal{R} = \{\{a,b\}, \{c\}, \{d,e\}\}$. If $A = \{a,b,d,e\}$, then $L_{\mathcal{R}}(A) = A, (L_{\mathcal{R}}(A))^{c} = \{c\}$. If $B = \{b,d,e\}$, then $\Phi_{\mathcal{P}}^{A}(B) = \{c\}$, and $\Phi_{\mathcal{P}}^{A}(\Phi_{\mathcal{P}}^{A}(B)) = \emptyset$. Hence, $\Phi_{\mathcal{P}}^{A}(B) \nsubseteq \Phi_{\mathcal{P}}^{A}(\Phi_{\mathcal{P}}^{A}(B))$.

Theorem 25 Let $\mathcal{P}$ be a primal on $(\boldsymbol{W}, \mathcal{R})$ associated with a subset A of $\boldsymbol{W}$. If $B \subseteq \boldsymbol{W}$, then

1) If $B \in \mathcal{P}$, then $\Phi_{\mathcal{P}}^{A}(B) = \emptyset$;

2) If $\mathcal{P} = 2^{\boldsymbol{W}} \backslash \boldsymbol{W}$, then $\Phi_{\mathcal{P}}^{A}(B) = \emptyset$.

Proof:

1) Suppose that $B \in \mathcal{P}$, then $B \backslash G \in \mathcal{P}$. Since $cl_{\mathcal{R}}^{A}(\emptyset) = \emptyset$, hence $\Phi_{\mathcal{P}}^{A}(B) = \bigcap \{G \subseteq \boldsymbol{W} \mid (B \backslash G) \in \mathcal{P}, cl_{\mathcal{R}}^{A}(G) = G\} = \emptyset$.
2) Obvious.

Definition 26 Let $(\boldsymbol{W}, \mathcal{R}, \mathcal{P})$ be a primal approximation space associated with $A \subseteq \boldsymbol{W}$. Then, for any $B \subseteq \boldsymbol{W}$, define the operators $cl_{\Phi_{\mathcal{P}}}^{A}, int_{\Phi_{\mathcal{P}}}^{A} : \boldsymbol{2^{W}} \to \boldsymbol{2^{W}}$ as follows:

$$cl_{\Phi_{\mathcal{P}}}^{A}(B) = B \cup \Phi_{\mathcal{P}}^{A}(B).$$

$$int_{\Phi_{\mathcal{P}}}^{A}(B) = B \cap (\Phi_{\mathcal{P}}^{A}(B^{c}))^{c}.$$

Proposition 27 Let $(\boldsymbol{W}, \mathcal{R}, \mathcal{P})$ be a primal approximation space associated with a set A of $\boldsymbol{W}$. Then, for any $B, \acute{B} \subseteq \boldsymbol{W}$, the following statements hold

1) $cl_{\Phi_{\mathcal{P}}}^{A}(\emptyset) = int_{\Phi_{\mathcal{P}}}^{A}(\emptyset)) = \emptyset$, *and* $cl_{\Phi_{\mathcal{P}}}^{A}(\boldsymbol{W}) = int_{\Phi_{\mathcal{P}}}^{A}(\boldsymbol{W})) = \boldsymbol{W}$.

2) $int_{\mathcal{R}}^{A}(B) \subseteq int_{\Phi_{\mathcal{P}}}^{A}(B) \subseteq B \subseteq cl_{\Phi_{\mathcal{P}}}^{A}(B) \subseteq cl_{\mathcal{R}}^{A}(B)$.

3) $cl_{\Phi_{P}}^{A}(B^{c}) = (int_{\Phi_{\mathcal{P}}}^{A}(B))^{c}$ *and* $int_{\Phi_{\mathcal{P}}}^{A}(B^{c}) = (cl_{\Phi_{\mathcal{P}}}^{A}(B))^{c}$.

4) $cl_{\Phi_{\mathcal{P}}}^{A}(\Phi_{\mathcal{P}}^{A}(B)) = \Phi_{\mathcal{P}}^{A}(B)$.

5) If $B \subseteq \acute{B}$, then $int_{\Phi_{\mathcal{P}}}^{A}(B) \subseteq int_{\Phi_{\mathcal{P}}}^{A}(\acute{B})$, and $cl_{\Phi_{\mathcal{P}}}^{A}(B) \subseteq cl_{\Phi_{\mathcal{P}}}^{A}(\acute{B})$.

6) $int_{\Phi_{\mathcal{P}}}^{A}(B) \cup int_{\Phi_{\mathcal{P}}}^{A}(\acute{B}) \subseteq int_{\Phi_{\mathcal{P}}}^{A}(B \cup \acute{B})$, and $int_{\Phi_{\mathcal{P}}}^{A}(B \cap \acute{B}) \subseteq int_{\Phi_{\mathcal{P}}}^{A}(B) \cap int_{\Phi_{\mathcal{P}}}^{A}(\acute{B})$.

7) $cl_{\Phi_{\mathcal{P}}}^{A}(B) \cup cl_{\Phi_{\mathcal{P}}}^{A}(\acute{B}) \subseteq cl_{\Phi_{\mathcal{P}}}^{A}(B \cup \acute{B})$, and $cl_{\Phi_{\mathcal{P}}}^{A}(B \cap \acute{B}) \subseteq cl_{\Phi_{\mathcal{P}}}^{A}(B) \cap cl_{\Phi_{\mathcal{P}}}^{A}(\acute{B})$.

8) $cl_{\Phi_{\mathcal{P}}}^{A}(B) = cl_{\Phi_{\mathcal{P}}}^{A} cl_{\Phi_{\mathcal{P}}}^{A}(B)$, and $int_{\Phi_{\mathcal{P}}}^{A} int_{\Phi_{\mathcal{P}}}^{A}(B) = int_{\Phi_{\mathcal{P}}}^{A}(B)$.

Proof:

Direct to prove.

Corollary 28 Let $(\boldsymbol{W}, \mathcal{R}, \mathcal{P})$ be a primal approximation space associated with a set A of $\boldsymbol{W}$. Then, the primal approximation operator $cl_{\Phi_{\mathcal{P}}}^{A}$ is a supra closure operator.

Definition 29 Let $(\boldsymbol{W}, \mathcal{R}, \mathcal{P})$ be a primal approximation space associated with a set A of $\boldsymbol{W}$. Then, the family

$$\boldsymbol{\Gamma_{\mathcal{P}}^{A}} = \{B^{c} \subseteq \boldsymbol{W} \mid B = cl_{\Phi_{\mathcal{P}}}^{A}(B)\}$$

is a supra topology on $\boldsymbol{W}$, named as primal supra topology on $\boldsymbol{W}$, induced by topology $\boldsymbol{T_{\mathcal{R}}^{A}}$ and primal $\mathcal{P}$.

According to (4) of Proposition 27, $\Phi_{\mathcal{P}}^{A}(B)$ is a supra closed set with respect to a primal supra topology $\Gamma_{\mathcal{P}}^{A}$ on $\boldsymbol{W}$.

The next example demonstrates that $\boldsymbol{\Gamma_{\mathcal{P}}^{A}}$ does not necessarily form a topology.

Example 30 Continued from Example 24**.** Obviously, $\boldsymbol{\tau_A} = \{\emptyset, \boldsymbol{W}, \{a, b, d, e\}\}$, $\boldsymbol{T_{\mathcal{R}}^A} = \{\emptyset, \boldsymbol{W}, \{a, b\}, \{d, e\}, \{a, b, d, e\}\}$, **and** $\boldsymbol{\Gamma_{\mathcal{P}}^A} = 2^{\boldsymbol{W}} \backslash \{\{c\}, \{a, c\}\}$. Note that {b, c}, {c, e} $\in \Gamma_{\mathcal{P}}^A$ but $\{c\} \notin \Gamma_{\mathcal{P}}^A$. As a result, $\boldsymbol{\Gamma_{\mathcal{P}}^A}$ is identified as a supra topology on $\boldsymbol{W}$**.**

Example 31 Continued Examples 19, 21. we get $\boldsymbol{\tau_A} = \{\emptyset, \boldsymbol{W}, \{a\}, \{b, c\}\}$, $\boldsymbol{T_{\mathcal{R}}^A} = \{\emptyset, \boldsymbol{W}, \{a\}\}$, $\boldsymbol{\Gamma_{\mathcal{Q}}^A} = 2^{\boldsymbol{W}}$ and $\boldsymbol{\Gamma_{\mathcal{P}}^A} = \{\emptyset, \boldsymbol{W}, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$.

Theorem 32 Let $(\boldsymbol{W}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{P}})$ be a primal approximation space associated with a set A of $\boldsymbol{W}$. Then, the primal supra topology $\boldsymbol{\Gamma_{\mathcal{P}}^A}$ is finer than $\boldsymbol{T_{\mathcal{R}}^A}$.

Proof:

Let $B^c \in \boldsymbol{T_{\mathcal{R}}^A}$. Then, $B = cl_{\mathcal{R}}^A(B)$. According to (1) of Proposition 17, $\Phi_{\mathcal{P}}^A(B) \subseteq B$. Thus, $cl_{\Phi_{\mathcal{P}}}^A(B) = B \cup \Phi_{\mathcal{P}}^A(B) = B$. This means that $B^c \in \boldsymbol{\Gamma_{\mathcal{P}}^A}$. So, $\boldsymbol{T_{\mathcal{R}}^A} \subseteq \boldsymbol{\Gamma_{\mathcal{P}}^A}$.

Theorem 33 Let $(\boldsymbol{W}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{P}})$ be a primal approximation space associated with a set A of $\boldsymbol{W}$. If $B \subseteq \boldsymbol{W}$, $\boldsymbol{\mathcal{P}} = 2^{\boldsymbol{W}} \backslash \boldsymbol{W}$, then $\boldsymbol{\Gamma_{\mathcal{P}}^A} = \boldsymbol{2^W}$.

Proof:

The proof is straightforward, as it follows directly from item (2) of Theorem 25.

Theorem 34 Let $\boldsymbol{\mathcal{P}}, \boldsymbol{Q}$ be two primals on $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$ associated with a subset A of $\boldsymbol{W}$. If $\boldsymbol{\mathcal{P}} \subseteq \boldsymbol{Q}$, then
1) $\Phi_{\mathcal{Q}}^A(B) \subseteq \Phi_{\mathcal{P}}^A(B)$.
2) $\Gamma_{\mathcal{P}}^A \subseteq \Gamma_{\mathcal{Q}}^A$.

Proof:

1) Suppose that $q \notin \Phi_{\mathcal{P}}^A(B)$, then there exists $G \subseteq \boldsymbol{W}$ with $q \notin G$, $(B \backslash G) \in \boldsymbol{\mathcal{P}}$, $cl_{\mathcal{R}}^A(G) = G$. Since $\boldsymbol{\mathcal{P}} \subseteq \boldsymbol{Q}$, then $q \notin G$, $(B \backslash G) \in \boldsymbol{Q}$, $cl_{\mathcal{R}}^A(G) = G$. Hence, $q \notin \Phi_{\mathcal{Q}}^A(B)$ and so $\Phi_{\mathcal{Q}}^A(B) \subseteq \Phi_{\mathcal{P}}^A(B)$.

2) Obvious from (1).

Definition 35 Let $(\boldsymbol{W}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{P}})$ be a primal approximation space associated with a set $\boldsymbol{A} \subseteq \boldsymbol{W}$**,** and $(\boldsymbol{\mathcal{C}}, \boldsymbol{S})$ be an approximation space associated with $\boldsymbol{E} \subseteq \boldsymbol{\mathcal{C}}$. A mapping *f:* $(\boldsymbol{W}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{P}}) \rightarrow (\boldsymbol{\mathcal{C}}, \boldsymbol{S})$ is named a primal approximation continuous, if

$$f^{-1}(int_{\boldsymbol{S}}^{\boldsymbol{E}}(\mathrm{U})) \subseteq int_{\Phi_{\mathcal{P}}}^{A}(f^{-1}(\mathrm{U})) \; \forall \mathrm{U} \subseteq \boldsymbol{\mathcal{C}}.$$

It is equivalent to the condition $cl_{\Phi_{\mathcal{P}}}^{A}\left(f^{-1}(\mathrm{U})\right) \subseteq f^{-1}(cl_{\boldsymbol{S}}^{\boldsymbol{E}}(\,(\mathrm{U})) \; \forall \mathrm{U} \subseteq \boldsymbol{\mathcal{C}}$**.**

Definition 36 Let $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$ be approximation space associated with a set $\boldsymbol{A} \subseteq \boldsymbol{W}$**,** and $(\boldsymbol{\mathcal{C}}, \boldsymbol{S}, \boldsymbol{Q})$ be a primal approximation space associated with $\boldsymbol{E} \subseteq \boldsymbol{\mathcal{C}}$. A mapping *f:* $(\boldsymbol{W}, \boldsymbol{\mathcal{R}}) \rightarrow (\boldsymbol{\mathcal{C}}, \boldsymbol{S}, \boldsymbol{Q})$ primal approximation open, if $f(int_{\boldsymbol{\mathcal{R}}}^{A}(\mathrm{M})) \subseteq int_{\Phi_{\boldsymbol{Q}}}^{\boldsymbol{E}}(f(\mathrm{M})) \; \forall \mathrm{M} \subseteq \boldsymbol{W}$**.**

Proposition 37 1) Every approximation continuous mapping is also primal approximation continuous.
2) Every approximation open mapping is also primal approximation open.

Proof:
The claims follow directly from part (2) of Proposition 27.

Definition 38 A primal approximation space $(\boldsymbol{W}, \boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{P}})$ associated with a set A of $\boldsymbol{W}$ is named:

1) $\boldsymbol{\mathcal{P}T_0}$**-space**, if for all $a, b \in A \; with \;\; a \neq b \;\; there \; exists \; a \; set \; M \subseteq \boldsymbol{W}$ such that

$$a \in int_{\Phi_{\mathcal{P}}}^{A}(M), and \; b \notin M.$$

2) $\boldsymbol{\mathcal{P}T_1}$**-space**, if for all$a, b \in A$, $with \;\; a \neq b \;\; there \; exist \; M, H \subseteq \boldsymbol{W}$ such that

$$a \in int_{\Phi_{\mathcal{P}}}^{A}(M), \;\; b \notin M \; and \;\; b \in int_{\Phi_{\mathcal{P}}}^{A}(H), \;\; a \notin H.$$

3) $\boldsymbol{\mathcal{P}T_2}$**-space**, if for all$a, b \in A$, $with \;\; a \neq b \;\; there \; exist \; M, H \subseteq \boldsymbol{W}$ such that

$$a \in int_{\Phi_{\mathcal{P}}}^{A}(M), \;\; b \in int_{\Phi_{\mathcal{P}}}^{A}(H), \;\; and \; M \cap H = \emptyset.$$

The relationships among the above axioms are as follows:

Lemma 39

1) $\mathcal{PT}_2$ -space ⇒ $\mathcal{PT}_1$ -space ⇒ $\mathcal{PT}_0$ -space.
2) Every $\boldsymbol{T}_{\mathfrak{w}}$ -space is $\mathcal{PT}_{\mathfrak{w}}$ -space, $\mathfrak{w} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$.

Proof:

1) Straightforward by using Definition 38.
2) According to (2) of Proposition 27, the proof is obvious.

The next examples shows that the converse of items of Lemma 39 does not hold as:

Example 40 Continued Examples **19.** Consider $\mathcal{P} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ is a primal on $\boldsymbol{W}$, then $\boldsymbol{\Gamma}_{\mathcal{P}}^{\boldsymbol{A}} = \{\emptyset, \boldsymbol{W}, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. It is a $\mathcal{PT}_0$ -space but not a $\mathcal{PT}_1$ -space.
Example 41 Continued Examples 31. If $\boldsymbol{Q} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$ is a primal on $\boldsymbol{W}$, then $\boldsymbol{\Gamma}_{\boldsymbol{Q}}^{\boldsymbol{A}} = \mathbf{2}^{\boldsymbol{W}}$.
Since $\boldsymbol{T}_{\mathcal{R}}^{\boldsymbol{A}} = \{\emptyset, \boldsymbol{W}, \{a\}\}$, then a primal approximation space $(\boldsymbol{W}, \mathcal{R}, \boldsymbol{Q})$ associated with a set A is $\boldsymbol{QT}_{\mathfrak{w}}$ but it is not $\boldsymbol{T}_{\mathfrak{w}}$, $\mathfrak{w} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$.
Theorem 42 Let $(\boldsymbol{W}, \mathcal{R}, \mathcal{P})$ be a primal approximation space associated with a set $\boldsymbol{A} \subseteq \boldsymbol{W}$, and $(\boldsymbol{C}, \boldsymbol{S})$ be an approximation space associated with $\boldsymbol{E} \subseteq \boldsymbol{C}$. If $f: (\boldsymbol{W}, \mathcal{R}, \mathcal{P}) \rightarrow (\boldsymbol{C}, \boldsymbol{S})$ is an injective primal approximation continuous mapping such that $f(\boldsymbol{A}) = \boldsymbol{E}$, and $(\boldsymbol{C}, \boldsymbol{S})$ is a $\boldsymbol{T}_{\mathfrak{w}}$ − space , then $(\boldsymbol{W}, \mathcal{R}, \mathcal{P})$ is a $\mathcal{PT}_{\mathfrak{w}}$ − space, $\mathfrak{w} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$.

Proof:
We prove this theorem for $\mathfrak{w} = \mathbf{2}$, and the rest of the proof is similar.
Let $a, b \in A$, $a \neq b$. Since f is an injective mapping, then $f(a) \neq f(b)$ in $\boldsymbol{E}$. According to $(\boldsymbol{C}, \boldsymbol{S})$ is a $\boldsymbol{T}_2$ − space, then there exist U, $V \subseteq \boldsymbol{C}$ such that $f(a) \in int_{\boldsymbol{S}}^{\boldsymbol{E}}(U), f(b) \in int_{\boldsymbol{S}}^{\boldsymbol{E}}(V), and\ U \cap V = \emptyset$. Hence, $a \in f^{-1}(int_{\boldsymbol{S}}^{\boldsymbol{E}}(U))$, $b \in f^{-1}(int_{\boldsymbol{S}}^{\boldsymbol{E}}(V))$. Since $f: (\boldsymbol{W}, \mathcal{R}, \mathcal{P}) \rightarrow (\boldsymbol{C}, \boldsymbol{S})$ is a primal approximation continuous mapping, then $f^{-1}(int_{\boldsymbol{S}}^{\boldsymbol{E}}(U)) \subseteq int_{\Phi_{\mathcal{P}}}^{\boldsymbol{A}}(f^{-1}(U))$ and $f^{-1}(int_{\boldsymbol{S}}^{\boldsymbol{E}}(V)) \subseteq int_{\Phi_{\mathcal{P}}}^{\boldsymbol{A}}(f^{-1}(V))$, which imply that $a \in int_{\Phi_{\mathcal{P}}}^{\boldsymbol{A}}(f^{-1}(U))$, and $b \in int_{\Phi_{\mathcal{P}}}^{\boldsymbol{A}}(f^{-1}(V))$. Choose $M = f^{-1}(U)$, $H = f^{-1}(V)$, so $(\boldsymbol{W}, \mathcal{R}, \mathcal{P})$ is a $\mathcal{PT}_2$ − space.

Theorem 43 Let $(\boldsymbol{W}, \mathcal{R})$ be approximation space associated with a set $\boldsymbol{A} \subseteq \boldsymbol{W}$, and $(\boldsymbol{C}, \boldsymbol{S}, \boldsymbol{Q})$ be a primal approximation space associated with $\boldsymbol{E} \subseteq \boldsymbol{C}$. If $f: (\boldsymbol{W}, \mathcal{R}) \rightarrow (\boldsymbol{C}, \boldsymbol{S}, \boldsymbol{Q})$ is a surjective primal approximation open mapping such that $f^{-1}(\boldsymbol{E}) = \boldsymbol{A}$ and $(\boldsymbol{W}, \mathcal{R})$ is a $\boldsymbol{T}_{\mathfrak{w}}$ − space , then $(\boldsymbol{C}, \boldsymbol{S}, \boldsymbol{Q})$ **is a** $\boldsymbol{QT}_{\mathfrak{w}}$ -space, $\mathfrak{w} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$.

Proof:
We prove this theorem for $\mathfrak{w} = \mathbf{2}$, and the rest of the proof is similar.
Let $s, t \in \boldsymbol{E}$, $s \neq t$, then $f^{-1}(\mathrm{s}) \neq f^{-1}(\mathrm{t})$ in $\boldsymbol{A}$. Since $(\boldsymbol{W}, \mathcal{R})$ is a $\boldsymbol{T}_2$ − space, then there exist $M, H \subseteq \boldsymbol{W}$ such that $f^{-1}(\mathrm{s}) \in int_{\mathcal{R}}^{\boldsymbol{A}}(M)$, $f^{-1}(\mathrm{t}) \in int_{\mathcal{R}}^{\boldsymbol{A}}(H), and\ M \cap H = \emptyset$. Since $f: (\boldsymbol{W}, \mathcal{R}) \rightarrow (\boldsymbol{C}, \boldsymbol{S}, \boldsymbol{Q})$ is a primal approximation open mapping, then $f(int_{\mathcal{R}}^{\boldsymbol{A}}(\mathrm{M})) \subseteq int_{\Phi_{\boldsymbol{Q}}}^{\boldsymbol{E}}(f(\mathrm{M}))$ and $f(int_{\mathcal{R}}^{\boldsymbol{A}}(\mathrm{H})) \subseteq int_{\Phi_{\boldsymbol{Q}}}^{\boldsymbol{E}}(f(\mathrm{H}))$, which imply that $s \in int_{\Phi_{\boldsymbol{Q}}}^{\boldsymbol{E}}(f(\mathrm{M}))$ and $t \in int_{\Phi_{\boldsymbol{Q}}}^{\boldsymbol{E}}(f(\mathrm{H}))$. Since f is a surjective mapping, then $f(\mathrm{M}) \cap f(\mathrm{H}) = \emptyset$. Consequently, $(\boldsymbol{C}, \boldsymbol{S}, \boldsymbol{Q})$ **is a** $\boldsymbol{QT}_2$ -space.

COMPUTATION OF THE PRIMAL SUPRA TOPOLOGY

The following algorithm (**Algorithm 1**) provides a structured procedure for computing the **primal supra topology** $\boldsymbol{\Gamma}_{\mathcal{P}}^{\boldsymbol{A}}$, a newly introduced framework that extends rough set approximation by integrating primal structures with supra-topological operators. The algorithm begins with the classical construction of equivalence classes and lower/upper approximations, then introduces a closure operator refined by primal sets, leading to the derivation of local and extended closure operators. By systematically determining which subsets satisfy primal closure conditions, the method constructs $\boldsymbol{\Gamma}_{\mathcal{P}}^{\boldsymbol{A}}$ as a family of sets that encapsulates the primal supra-topological structure over a universe $\boldsymbol{W}$.

Providing this explicit algorithm is essential, since the complexity of primal approximation methods makes replication difficult when only theoretical definitions are given. By laying out clear computational steps, including the handling of empty and non-empty sets, this algorithm ensures reproducibility and transparency in implementation. It also highlights the potential for further applications, as the primal supra topology captures nuanced relationships between rough approximations and topological closure systems. The availability of this detailed algorithmic flow lowers the barrier for replication, comparison, and adaptation in future studies, especially where code availability may be limited.

Algorithm 1: Computation of Primal Supra Topology $\boldsymbol{\Gamma_{\mathcal{P}}^{A}}$

Input:

- $\boldsymbol{W}$: A finite or infinite universe.
- $\boldsymbol{\mathcal{R}}$: An equivalence relation on $\boldsymbol{W}$.
- $A \subseteq \boldsymbol{W}$: A subset of $\boldsymbol{W}$.
- $\boldsymbol{\mathcal{P}} \subseteq 2^{\boldsymbol{W}}$: A primal set of subsets of $\boldsymbol{W}$.

Output: The family $\boldsymbol{\Gamma_{\mathcal{P}}^{A}}$, representing the primal supra topology on $\boldsymbol{W}$.

Algorithm Steps:

Step 1: Compute Equivalence Classes

For each $x \in \boldsymbol{W}$, compute its equivalence class under $\boldsymbol{\mathcal{R}}$: $[x]_{\boldsymbol{\mathcal{R}}} = \{y \in \boldsymbol{W} : (x, y) \in \boldsymbol{\mathcal{R}}\}$.

Store these classes for later use.

Step 2: Compute Lower and Upper Approximations

For the subset $A \subseteq \boldsymbol{W}$:

- Lower Approximation $L_{\boldsymbol{\mathcal{R}}}(A)$:

$$L_{\boldsymbol{\mathcal{R}}}(A) = \bigcup_{x \in \boldsymbol{W}} \{[x]_{\boldsymbol{\mathcal{R}}} : [x]_{\boldsymbol{\mathcal{R}}} \subseteq A\}$$

- Upper Approximation $U_{\boldsymbol{\mathcal{R}}}(A)$:

$$U_{\boldsymbol{\mathcal{R}}}(A) = \bigcup_{x \in \boldsymbol{W}} \{[x]_{\boldsymbol{\mathcal{R}}} : [x]_{\boldsymbol{\mathcal{R}}} \cap A \neq \emptyset\}$$

Step 3: Define the Closure Operator $cl_{\boldsymbol{\mathcal{R}}}^{A}(B)$

For any subset $B \subseteq \boldsymbol{W}$:

- If $B \neq \emptyset$: $cl_{\boldsymbol{\mathcal{R}}}^{A}(B) = (L_{\boldsymbol{\mathcal{R}}}(A))^{c} \cup U_{\boldsymbol{\mathcal{R}}}(B)$
- If $B = \emptyset$: $cl_{\boldsymbol{\mathcal{R}}}^{A}(B) = \emptyset$

Step 4: Compute the Local Set $\Phi_{\mathcal{P}}^{A}(B)$

For any subset $B \subseteq \boldsymbol{W}$:

- Compute the collection of sets: $\mathfrak{D} = \{G \subseteq \boldsymbol{W} \mid (B \backslash G) \in \boldsymbol{\mathcal{P}}, cl_{\boldsymbol{\mathcal{R}}}^{A}(G) = G\}$.

Ensure that $\mathfrak{D}$ is non-empty.

- Compute the intersection of all sets in $\mathfrak{D}$:

$$\Phi_{\mathcal{P}}^{A}(B) = \bigcap \mathfrak{D}.$$

If $\mathfrak{D}$ is empty, then $\Phi_{\mathcal{P}}^{A}(B) = \emptyset$.

Step 5: Compute the Extended Closure Operator

Using the local set $\Phi_{\mathcal{P}}^{A}(B)$, define the extended closure operator:

$$cl_{\Phi_{\mathcal{P}}}^{A}(B) = B \cup \Phi_{\mathcal{P}}^{A}(B).$$

Step 6: Determine Membership in $\boldsymbol{\Gamma_{\mathcal{P}}^{A}}$

For each $B \subseteq \boldsymbol{W}$:

- If $B = cl_{\Phi_{\mathcal{P}}}^{A}(B)$, include $B^{c} \in \Gamma_{\mathcal{P}}^{A}$.
- Otherwise, exclude B^{c}.

Step 7: Return $\boldsymbol{\Gamma_{\mathcal{P}}^{A}}$

Output:

$$\Gamma_{\mathcal{P}}^{A} = \{B^{c} \subseteq \boldsymbol{W} \mid B = cl_{\Phi_{\mathcal{P}}}^{A}(B)\}$$

End

The algorithm provides a constructive procedure for deriving the primal supra topology $\boldsymbol{\Gamma_{\mathcal{P}}^{A}}$ from a given approximation space $(\boldsymbol{W}, \boldsymbol{\mathcal{R}})$, a subset $A \subseteq \boldsymbol{W}$, and a primal set $\boldsymbol{\mathcal{P}} \subseteq 2^{\boldsymbol{W}}$. The steps formalize the computation of

equivalence classes, rough approximations, closure operators, and primal extensions, yielding $\boldsymbol{\Gamma}_{\mathcal{P}}^{\mathbf{A}}$ as a family of subsets satisfying primal supra-topological closure conditions.

Conclusions

The study presents a comprehensive framework for primal approximation spaces, expanding the theoretical boundaries of rough set theory. By defining primal structures and their associated operators, we establish a robust foundation for integrating topological and algebraic concepts within rough approximations. Theorems and examples underscore the practical implications of these developments, showcasing their potential in refining decision-making processes and enhancing computational models.

The novel approaches presented, including the application of **Super HyperTopology** and **Neutrosophic Super HyperTopology**, underline the versatility of the methods in addressing real-world problems. Furthermore, this work bridges the gap between classical rough set models and advanced topological frameworks, offering a richer understanding of their potential applications. The results demonstrate that primal approximation spaces not only extend the scope of traditional rough sets but also provide new perspectives on the interplay between topology and approximation theory. In addition, the proposed algorithm for computing the primal supra topology $\boldsymbol{\Gamma}_{\mathcal{P}}^{\mathbf{A}}$ offers a direct and reproducible procedure for implementing the theoretical framework. By formalizing equivalence classes, rough approximations, and closure operators within a primal structure, the algorithm provides a clear computational pathway that supports transparency and facilitates future replication. Collectively, these advancements contribute to the versatility of rough sets in addressing diverse challenges across disciplines.

Future Research Directions

The exploration of primal approximation spaces opens numerous avenues for future research:

1. **Advanced Topological Structures**: Investigate the integration of more complex topological constructs, such as hyper-connected or multifaceted topologies, within primal approximation spaces.
2. **Applications in Artificial Intelligence**: Apply primal approximation frameworks to machine learning, knowledge discovery, and expert systems, particularly in handling incomplete or inconsistent data.
3. **Hybrid Models**: Develop hybrid models combining primal approximation spaces with other mathematical frameworks, such as fuzzy sets or intuitionistic sets, to address more nuanced real-world problems.
4. **Computational Algorithms**: Design efficient algorithms for implementing primal approximation operators in large-scale datasets, ensuring their scalability and computational feasibility.
5. **Interdisciplinary Studies**: Expand applications to fields such as bioinformatics, economics, and social network analysis, leveraging primal structures for innovative solutions to domain-specific problems such as [10-13].
6. **Integration of Novel Topologies**: Explore the utility of **Super HyperTopology** and **Neutrosophic Super HyperTopology** in solving complex, interdisciplinary challenges.

These directions aim to harness the full potential of primal approximation spaces, fostering their adoption in both theoretical and practical contexts.

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Chapter 12

Lukasiewicz fuzzy hyper hoop subalgebras and filters in hyper hoops

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ABSTRACT

We introduced the notion of Łukasiewicz *f*uzzy ḥyper ḥoop subalgebra and Łukasiewicz *f*uzzy *f*ilters in ḥyper ḥoops by applying the Łukasiewicz *f*uzzy set to ḥyper ḥoop subalgebra and *f*ilter of ḥyper ḥoops. We provided the conditions for the Łukasiewicz *f*uzzy set to be a Łukasiewicz *f*uzzy ḥyper ḥoop subalgebra and Łukasiewicz *f*uzzy *f*ilter of ḥyper ḥoops.

Keywords: Łukasiewicz *f*uzzy ḥyper ḥoop subalgebra, Łukasiewicz *f*uzzy *f*ilters, Ḥyper ḥoops, Łukasiewicz *f*uzzy set, Ḟuzzy point.

INTRODUCTION

Ḥoop is an algebraic structure introduced by B. Bosbach. The Ḥyper algebraic structure theory was introduced by Marty (1934) and studied by many researchers [6],[7],[8]. As a generalization of ḥoop, Borzooei et.al explore the ḥyper structure on ḥoops and introduced ḥyper ḥoop algebra [4]. R.A.Borzooei studied the concept of *f*uzzy *f*ilters and *f*uzzy subalgebras on ḥyper ḥoops [3].Łukasiewicz logic is the non-classical many valued logic. In recent years, many researchers studied the Łukasiewicz *f*uzzy logic in various algebraic structures [1],[5],[9]. M.Mohseni Takallo explored Łukasiewicz *f*uzzy *f*ilters on ḥoop algebras [9].

In this chapter, our focus is to apply the Łukasiewicz *f*uzzy set to the structure of ḥyper ḥoops. The chapter is organized as follows: In section 2, we introduced some basic concepts. In section 3, Łukasiewicz *f*uzzy ḥyper ḥoop subalgebra is introduced and in section 4, Łukasiewicz *f*uzzy *f*ilter of ḥyper ḥoops is studied.

PRELIMINARIES

Definition 1 [3] A non-empty set $(\mathbb{H},\circledast,\rightsquigarrow,e)$ satisfying the following conditions is said to be binary a Ḥyper ḥoop, where $\circledast$ and $\rightsquigarrow$ are the binary ḥyper operations.

$(\mathbb{H}\mathbb{H}1)$ $(\mathbb{H},\circledast,e)$ is a commutative semiḥypergroup

$(\mathbb{H}\mathbb{H}2)$ $e \in \mathfrak{j} \rightsquigarrow \mathfrak{j}$

$(\mathbb{H}\mathbb{H}3)$ $(\mathfrak{j} \rightsquigarrow \mathfrak{r}) \circledast \mathfrak{j} = (\mathfrak{r} \rightsquigarrow \mathfrak{j}) \circledast \mathfrak{r}$

$(\mathbb{H}\mathbb{H}4)$ $\mathfrak{j} \rightsquigarrow (\mathfrak{r} \rightsquigarrow \mathfrak{y}) = (\mathfrak{j} \circledast \mathfrak{r}) \rightsquigarrow \mathfrak{y}$

$(\mathbb{H}\mathbb{H}5)$ $e \in \mathfrak{j} \rightsquigarrow e$

$(\mathbb{H}\mathbb{H}6)$ $e \in \mathfrak{j} \rightsquigarrow \mathfrak{r}$, $e \in \mathfrak{r} \rightsquigarrow \mathfrak{j}$ implies $\mathfrak{j} = \mathfrak{r}$

$(\mathbb{H}\mathbb{H}7)$ $e \in \mathfrak{j} \rightsquigarrow \mathfrak{r}$, $e \in \mathfrak{r} \rightsquigarrow \mathfrak{y}$ implies $e \in \mathfrak{j} \rightsquigarrow \mathfrak{y}$ for all $\mathfrak{j},\mathfrak{r},\mathfrak{y} \in \mathbb{H}$

On ḥyper ḥoop $\mathbb{H}$, we define $\mathfrak{j} \leq \mathfrak{r}$ if and only if $e \in \mathfrak{j} \rightsquigarrow \mathfrak{r}$. If there is an element $0 \in \mathbb{H}$ such that $0 \leq \mathfrak{j}$ for all $\mathfrak{j} \in \mathbb{H}$, then the ḥyper ḥoop is said to be bounded.

If $\mathfrak{D}$ and $\mathfrak{T}$ are subsets of $\mathbb{H}$, we define $\mathfrak{D} \ll \mathfrak{T}$ iff there exists $\mathfrak{d} \in \mathfrak{D}$ and $\mathfrak{t} \in \mathfrak{T}$ such that $\mathfrak{d} \leq \mathfrak{t}$ and define $\mathfrak{D} \leq \mathfrak{T}$ iff for any $\mathfrak{d} \in \mathfrak{D}$ there exists $\mathfrak{t} \in \mathfrak{T}$ such that $\mathfrak{d} \leq \mathfrak{t}$.

Proposition 2[3] Every ḥyper ḥoop satisfies the following conditions for all $\mathfrak{j},\mathfrak{r},\mathfrak{y} \in \mathbb{H}$ and $\mathfrak{D},\mathfrak{T},\mathfrak{Z} \subseteq \mathbb{H}$

$(\mathbb{H}\mathbb{H}8)$ $\mathfrak{j} \circledast \mathfrak{r} \ll \mathfrak{y} \Leftrightarrow \mathfrak{j} \leq \mathfrak{r} \rightsquigarrow \mathfrak{y}$

$(\mathbb{H}\mathbb{H}9)$ $\mathfrak{D} \circledast \mathfrak{T} \ll \mathfrak{Z} \Leftrightarrow \mathfrak{D} \leq \mathfrak{T} \rightsquigarrow \mathfrak{Z}$

$(\mathbb{H}\mathbb{H}10)$ $\mathfrak{j} \in e \rightsquigarrow \mathfrak{j}$

$(\mathbb{H}\mathbb{H}11)$ $\mathfrak{y} \rightsquigarrow \mathfrak{r} \leq (\mathfrak{r} \rightsquigarrow \mathfrak{j}) \rightsquigarrow (\mathfrak{y} \rightsquigarrow \mathfrak{j})$

$(\mathbb{H}\mathbb{H}12)$ $\mathfrak{y} \rightsquigarrow \mathfrak{r} \ll (\mathfrak{j} \rightsquigarrow \mathfrak{y}) \rightsquigarrow (\mathfrak{j} \rightsquigarrow \mathfrak{r})$

$(\mathbb{H}\mathbb{H}13)$ $e \circledast e = \{e\}$

$(\mathbb{H}\mathbb{H}14)$ $\mathfrak{r} \leq \mathfrak{j} \rightsquigarrow \mathfrak{r}$

Definition 3 [3] A non-empty subset $\dot{\mathbb{F}}$ of $\mathbb{H}$ is a ḥyper ḥoop-subalgebra of $\mathbb{H}$ if $\dot{\mathbb{F}}$ is a ḥyper ḥoop with the binary operations $\circledast$ and $\rightsquigarrow$ on $\mathbb{H}$.

Definition 4 [3] A non-empty subset $\dot{\mathbb{F}}$ of $\mathbb{H}$ satisfying the following conditions is said to be a $f$ilter of $\mathbb{H}$.

i. $\mathfrak{j} \in \dot{\mathbb{F}}$ and $\mathfrak{j} \leq \mathfrak{r}$ imply $\mathfrak{r} \in \dot{\mathbb{F}}$, $\quad \forall \mathfrak{j},\mathfrak{r} \in \mathbb{H}$
ii. $\mathfrak{j} \circledast \mathfrak{r} \subseteq \mathbb{H}$

Definition 5 [9] A $f$uzzy set ξ is said to be a $f$uzzy point with support $\mathfrak{j}$ and value α and is denoted by $\langle \mathfrak{j}/\alpha \rangle$ if it is of the form

$$\xi(\mathfrak{r}) = \begin{cases} \alpha & if\ \mathfrak{r} = \mathfrak{j} \\ 0 & if\ \mathfrak{r} \neq \mathfrak{j} \end{cases}$$

A $f$uzzy point $\langle \mathfrak{j}/\alpha \rangle$ is said to be contained in a $f$uzzy set ξ if $\xi(\mathfrak{j}) \geq \alpha$, denoted by $\langle \mathfrak{j}/\alpha \rangle \in \xi$.

Definition 6 [3] If a $f$uzzy subset ξ of $\mathbb{H}$ meets the following conditions, then it is said to be the $f$uzzy ḥyper ḥoop subalgebra of $\mathbb{H}$.

i. $\inf \xi(\mathfrak{j} \circledast \mathfrak{r}) \geq min\{\xi(\mathfrak{j}), \xi(\mathfrak{r})\}$
ii. $\inf \xi(\mathfrak{j} \rightsquigarrow \mathfrak{r}) \geq min\{\xi(\mathfrak{j}), \xi(\mathfrak{r})\}$

Definition 7 [3] If a $f$uzzy set ξ of $\mathbb{H}$ meets the following conditions, then it is said to be the $f$uzzy $f$ilter of $\mathbb{H}$.

i. $\xi(e) \geq \xi(\mathfrak{j})$

ii. $\xi(\mathfrak{r}) \geq min\{\xi(\mathfrak{j}), \sup \xi(\mathfrak{j} \rightsquigarrow \mathfrak{r})\}$

Definition 8 [9] Let ξ be a $f$uzzy set in $\mathbb{H}$. A Łukasiewicz $f$uzzy set L_ξ^θ of ξ in $\mathbb{H}$ is a function $L_\xi^\theta: \mathbb{H} \to [0,1]$ defined by $L_\xi^\theta(\mathfrak{j}) = max\{0, \xi(\mathfrak{j}) + \theta - 1\}$ where $\theta \in (0,1)$.

Lukasiewicz $f$uzzy ḥyper ḥoop subalgebra

Definition 9 Let ξ be a $f$uzzy subset of $\mathbb{H}$. Then its Łukasiewicz $f$uzzy set L_ξ^θ is called a Łukasiewicz $f$uzzy ḥyper ḥoop subalgebra of $\mathbb{H}$, if it satisfies the following conditions for all $\mathfrak{j}, \mathfrak{r} \in \mathbb{H}$

i. $inf_{\mathfrak{y} \in \mathfrak{j} \circledast \mathfrak{r}} L_\xi^\theta(\mathfrak{y}) \geq min\{L_\xi^\theta(\mathfrak{j}), L_\xi^\theta(\mathfrak{r})\}$ (1)

ii. $inf_{\mathfrak{y} \in \mathfrak{j} \rightsquigarrow \mathfrak{r}} L_\xi^\theta(\mathfrak{y}) \geq min\{L_\xi^\theta(\mathfrak{j}), L_\xi^\theta(\mathfrak{r})\}$ (2)

Theorem 10 Let ξ be a $f$uzzy subset of $\mathbb{H}$. Then its Łukasiewicz $f$uzzy set L_ξ^θ of $\mathbb{H}$ is a Łukasiewicz $f$uzzy ḥyper ḥoop subalgebra of $\mathbb{H}$, iff for any $\mathfrak{j}, \mathfrak{r} \in \mathbb{H}$ and $\alpha, \beta \in (0,1]$,

$\langle \mathfrak{j}/\alpha \rangle \in L_\xi^\theta, \langle \mathfrak{r}/\beta \rangle \in L_\xi^\theta \Rightarrow \langle \mathfrak{y}/min\{\alpha, \beta\} \rangle \in L_\xi^\theta$ for all $\mathfrak{y} \in \mathfrak{j} \circledast \mathfrak{r}$ and $\mathfrak{y} \in \mathfrak{j} \rightsquigarrow \mathfrak{r}$ (3)

Proof: Suppose that L_ξ^θ is a Łukasiewicz $f$uzzy ḥyper ḥoop subalgebra of $\mathbb{H}$.

Let $\mathfrak{j}, \mathfrak{r} \in \mathbb{H}$ and $\alpha, \beta \in (0,1]$ be such that $\langle \mathfrak{j}/\alpha \rangle \in L_\xi^\theta$ and $\langle \mathfrak{r}/\beta \rangle \in L_\xi^\theta$.

Then $L_\xi^\theta(\mathfrak{j}) \geq \alpha$ and $L_\xi^\theta(\mathfrak{r}) \geq \beta$. By the definition of Łukasiewicz $f$uzzy ḥyper ḥoop subalgebra, we have that

$inf_{\mathfrak{y} \in \mathfrak{j} \circledast \mathfrak{r}} L_\xi^\theta(\mathfrak{y}) \geq min\{L_\xi^\theta(\mathfrak{j}), L_\xi^\theta(\mathfrak{r})\}$ and $inf_{\mathfrak{y} \in \mathfrak{j} \rightsquigarrow \mathfrak{r}} L_\xi^\theta(\mathfrak{y}) \geq min\{L_\xi^\theta(\mathfrak{j}), L_\xi^\theta(\mathfrak{r})\}$ for all $\mathfrak{j}, \mathfrak{r} \in \mathbb{H}$. This leads to

$$inf_{\mathfrak{y} \in \mathfrak{j} \circledast \mathfrak{r}} L_\xi^\theta(\mathfrak{y}) \geq min\{L_\xi^\theta(\mathfrak{j}), L_\xi^\theta(\mathfrak{r})\} \geq min\{\alpha, \beta\} \text{ and}$$

$$inf_{\mathfrak{y} \in \mathfrak{j} \rightsquigarrow \mathfrak{r}} L_\xi^\theta(\mathfrak{y}) \geq min\{L_\xi^\theta(\mathfrak{j}), L_\xi^\theta(\mathfrak{r})\} \geq min\{\alpha, \beta\}.$$

Hence $L_\xi^\theta(\mathfrak{y}) \geq min\{\alpha, \beta\}$ for all $\mathfrak{y} \in \mathfrak{j} \circledast \mathfrak{r}$ and $L_\xi^\theta(\mathfrak{y}) \geq min\{\alpha, \beta\}$ for all $\mathfrak{y} \in \mathfrak{j} \rightsquigarrow \mathfrak{r}$, which implies that

$$\langle \mathfrak{y}/min\{\alpha, \beta\} \rangle \in L_\xi^\theta \text{ for all } \mathfrak{y} \in \mathfrak{j} \circledast \mathfrak{r} \text{ and } \mathfrak{y} \in \mathfrak{j} \rightsquigarrow \mathfrak{r}.$$

Conversely suppose that the condition (3) holds.

We know that $\langle \mathfrak{j}/L_\xi^\theta(\mathfrak{j}) \rangle \in L_\xi^\theta$ and $\langle \mathfrak{r}/L_\xi^\theta(\mathfrak{r}) \rangle \in L_\xi^\theta$ for all $\mathfrak{j}, \mathfrak{r} \in \mathbb{H}$. It follows from (3) that

$\langle \mathfrak{y}/min\{L_\xi^\theta(\mathfrak{j}), L_\xi^\theta(\mathfrak{r})\} \rangle \in L_\xi^\theta$ for all $\mathfrak{y} \in \mathfrak{j} \circledast \mathfrak{r}$ and $\mathfrak{y} \in \mathfrak{j} \rightsquigarrow \mathfrak{r}$.

i.e., $L_\xi^\theta(\mathfrak{y}) \geq min\{L_\xi^\theta(\mathfrak{j}), L_\xi^\theta(\mathfrak{r})\}$ for all $\mathfrak{y} \in \mathfrak{j} \circledast \mathfrak{r}$ and

$L_\xi^\theta(\mathfrak{y}) \geq min\{L_\xi^\theta(\mathfrak{j}), L_\xi^\theta(\mathfrak{r})\}$ for all $\mathfrak{y} \in \mathfrak{j} \rightsquigarrow \mathfrak{r}$.

And so we have $inf_{\mathfrak{y} \in \mathfrak{j} \circledast \mathfrak{r}} L_\xi^\theta(\mathfrak{y}) \geq min\{L_\xi^\theta(\mathfrak{j}), L_\xi^\theta(\mathfrak{r})\}$ and $inf_{\mathfrak{y} \in \mathfrak{j} \rightsquigarrow \mathfrak{r}} L_\xi^\theta(\mathfrak{y}) \geq min\{L_\xi^\theta(\mathfrak{j}), L_\xi^\theta(\mathfrak{r})\}$.

Therefore, L_ξ^θ of $\mathbb{H}$ is a Łukasiewicz $f$uzzy ḥyper ḥoop subalgebra of $\mathbb{H}$.

Example 11 Let $\mathbb{H} = \{a, e\}$ be a set with identity element 'e' and ḥyper operations $\circledast$ and $\rightsquigarrow$.

$\circledast$	e	a
e	$\{e\}$	$\{e, a\}$
a	$\{e, a\}$	$\{a\}$

$\rightsquigarrow$	e	a
e	$\{e, a\}$	$\{a\}$
a	$\{e\}$	$\{e, a\}$

Clearly $(\mathbb{H},\circledast,\rightsquigarrow,e)$ is a bounded ḥyper ḥoop.

Define a $f$uzzy set $\xi:\mathbb{H}\to[0,1]$ by,

$$\xi=\frac{0.632}{a}+\frac{0.479}{e}$$

Then for $\theta=0.837$, the Łukasiewicz $f$uzzy set L_ξ^θ in $\mathbb{H}$ is given by,

$$L_\xi^\theta=\frac{0.469}{a}+\frac{0.316}{e}$$

We can verify that L_ξ^θ in $\mathbb{H}$ is a Łukasiewicz $f$uzzy ḥyper ḥoop subalgebra of $\mathbb{H}$.

Example 12 For the bounded ḥyper ḥoop $(\mathbb{H},\circledast,\rightsquigarrow,e)$ in example 11,

Define the $f$uzzy set $\xi:\mathbb{H}\to[0,1]$ as follows:

$$\xi=\frac{0.315}{a}+\frac{0.726}{e}$$

Then for $\theta=0.681$, the Łukasiewicz $f$uzzy set L_ξ^θ in $\mathbb{H}$ is given by,

$$L_\xi^\theta=\frac{0}{a}+\frac{0.407}{e}$$

Since $inf_{\mathfrak{y}\in e\rightsquigarrow e}L_\xi^\theta(\mathfrak{y})\not\geq min\{L_\xi^\theta(e),L_\xi^\theta(e)\}$, L_ξ^θ in $\mathbb{H}$ is not a Łukasiewicz $f$uzzy ḥyper ḥoop subalgebra of $\mathbb{H}$.

Lukasiewicz *f*uzzy *f*ilter of ḥyper ḥoops

Definition 13 Let ξ be a $f$uzzy subset of $\mathbb{H}$. Then its Łukasiewicz $f$uzzy set L_ξ^θ is called a Łukasiewicz $f$uzzy $f$ilter of $\mathbb{H}$, if it satisfies the following conditions for all $\mathfrak{j},\mathfrak{r}\in\mathbb{H}$

i. $L_\xi^\theta(e)\geq L_\xi^\theta(\mathfrak{j})$ (4)

ii. $L_\xi^\theta(\mathfrak{r})\geq min\{L_\xi^\theta(\mathfrak{j}),sup_{\mathfrak{y}\in\mathfrak{j}\rightsquigarrow\mathfrak{r}}L_\xi^\theta(\mathfrak{y})\}$ (5)

Theorem 14 Let ξ be a $f$uzzy subset of $\mathbb{H}$. Then its Łukasiewicz $f$uzzy set L_ξ^θ of $\mathbb{H}$ is a Łukasiewicz $f$uzzy $f$ilter of $\mathbb{H}$, iff for any $\mathfrak{j},\mathfrak{r}\in\mathbb{H}$ and $\alpha,\beta\in(0,1]$,

i. $\langle\mathfrak{j}/\alpha\rangle\in L_\xi^\theta\ \Rightarrow\ \langle e/\alpha\rangle\in L_\xi^\theta$ (6)

ii. $\langle\mathfrak{j}/\alpha\rangle\in L_\xi^\theta$, $\langle\mathfrak{z}/\beta\rangle\in L_\xi^\theta$ where $\mathfrak{z}\in\mathfrak{j}\rightsquigarrow\mathfrak{r}$ and $L_\xi^\theta(\mathfrak{z})=\sup L_\xi^\theta(\mathfrak{j}\rightsquigarrow\mathfrak{r})$ implies $\langle\mathfrak{r}/min\{\alpha,\beta\}\rangle\in L_\xi^\theta$ (7)

Proof: Assume that L_ξ^θ is a Łukasiewicz $f$uzzy $f$ilter of $\mathbb{H}$.

Let $\mathfrak{j}\in\mathbb{H}$ and $\alpha\in(0,1]$ be such that $\langle\mathfrak{j}/\alpha\rangle\in L_\xi^\theta$

Then $L_\xi^\theta(\mathfrak{j})\geq\ \alpha$. By using (4), we have that $L_\xi^\theta(e)\geq L_\xi^\theta(\mathfrak{j})\geq\ \alpha$.

$\Rightarrow\langle e/\alpha\rangle\in L_\xi^\theta$. Thus (6) is proved.

Let $\langle\mathfrak{j}/\alpha\rangle\in L_\xi^\theta$ and $\langle\mathfrak{z}/\beta\rangle\in L_\xi^\theta$ where $\mathfrak{z}\in\mathfrak{j}\rightsquigarrow\mathfrak{r}$ and $L_\xi^\theta(\mathfrak{z})=\sup L_\xi^\theta(\mathfrak{j}\rightsquigarrow\mathfrak{r})$. Then $L_\xi^\theta(\mathfrak{j})\geq\ \alpha$ and

$supL_\xi^\theta(\mathfrak{j}\rightsquigarrow\mathfrak{r})=L_\xi^\theta(\mathfrak{z})\geq\ \beta$. The condition (5) gives that,

$L_\xi^\theta(\mathfrak{r}) \geq min\{L_\xi^\theta(\mathfrak{j}), \sup\ L_\xi^\theta(\mathfrak{j} \rightsquigarrow \mathfrak{r})\} \geq min\{\alpha, \beta\}$. This implies $\langle \mathfrak{r}/min\{\alpha, \beta\}\rangle \in L_\xi^\theta$. This proves (7).

Suppose that L_ξ^θ satisfies (6) and (7). Since $\langle \mathfrak{j}/L_\xi^\theta(\mathfrak{j})\rangle \in L_\xi^\theta$ for all $\mathfrak{j} \in \mathbb{H}$, we have $\langle e/L_\xi^\theta(\mathfrak{j})\rangle \in L_\xi^\theta$ by (6).

$L_\xi^\theta(e) \geq L_\xi^\theta(\mathfrak{j})$ for all $\mathfrak{j} \in \mathbb{H}$.

We have $\langle \mathfrak{j}/L_\xi^\theta(\mathfrak{j})\rangle \in L_\xi^\theta$ and $\langle \mathfrak{z}/L_\xi^\theta(\mathfrak{z})\rangle \in L_\xi^\theta$, $\mathfrak{z} \in \mathfrak{j} \rightsquigarrow \mathfrak{r}$ and $L_\xi^\theta(\mathfrak{z}) = \sup L_\xi^\theta(\mathfrak{j} \rightsquigarrow \mathfrak{r})$. Then by using (7)

$$\langle \mathfrak{r}/min\{L_\xi^\theta(\mathfrak{j}), L_\xi^\theta(\mathfrak{z})\}\rangle \in L_\xi^\theta$$

$$\langle \mathfrak{r}/min\{L_\xi^\theta(\mathfrak{j}), \sup L_\xi^\theta(\mathfrak{j} \rightsquigarrow \mathfrak{r})\}\rangle \in L_\xi^\theta$$

$$\Rightarrow L_\xi^\theta(\mathfrak{r}) \geq min\{L_\xi^\theta(\mathfrak{j}), \sup L_\xi^\theta(\mathfrak{j} \rightsquigarrow \mathfrak{r})\}$$

Hence, L_ξ^θ is a Łukasiewicz $f$uzzy $f$ilter of ḥyper ḥoops.

Example 15 Let $\mathbb{H} = \{e, a, b\}$ be a set with identity element 'e' and ḥyper operations $\circledast$ and $\rightsquigarrow$.

$\circledast$	e	a	b
e	$\{e\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{a, b\}$	$\{a, b\}$
b	$\{b\}$	$\{a, b\}$	$\{b\}$

$\rightsquigarrow$	e	a	b
e	$\{e\}$	$\{a\}$	$\{b\}$
a	$\{e\}$	$\{e, a, b\}$	$\{e, b\}$
b	$\{e\}$	$\{a\}$	$\{e, b\}$

Clearly $(\mathbb{H}, \circledast, \rightsquigarrow, e)$ is a bounded ḥyper ḥoop.

Define a $f$uzzy set $\xi : \mathbb{H} \to [0,1]$ by,

$$\xi = \frac{1}{e} + \frac{0.6}{a} + \frac{0.8}{a}$$

Then for $\theta = 0.5$, the Łukasiewicz $f$uzzy set L_ξ^θ in $\mathbb{H}$ is given by,

$$L_\xi^\theta = \frac{0.5}{e} + \frac{0.1}{a} + \frac{0.3}{a}$$

We can verify that L_ξ^θ in $\mathbb{H}$ is a Łukasiewicz $f$uzzy $f$ilter of $\mathbb{H}$.

Conclusions

In this chapter, we applied the Łukasiewicz $f$uzzy set to the structure of ḥyper ḥoops. We introduced the concept of Łukasiewicz $f$uzzy ḥyper ḥoop subalgebra and Łukasiewicz $f$uzzy $f$ilter of ḥyper ḥoops. The conditions for the Łukasiewicz $f$uzzy set to be a Łukasiewicz $f$uzzy ḥyper ḥoop subalgebra and Łukasiewicz $f$uzzy $f$ilter of ḥyper ḥoops is provided.

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Chapter 13

Single-valued Łukasiewicz Neutrosophic Super Hyper Set and its Applications

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ABSTRACT

This chapter deals with the concept of Neutrosophic in Hyper and Super Hyper set and the discussion of real-world problem with sets of multiple neutrosophic components for defining the problem more precisely. Also, we apply these concepts together with Łukasiewicz theory for the reduction and elimination of indetermined or false values obtained in the data. Moreover, we have provided an example and discussed it for every concept of this study.

Keywords: Single-valued Neutrosophic hyper set, Single-valued Neutrosophic Super hyper set, Single-valued Łukasiewicz Neutrosophic set, Single-valued Łukasiewicz Neutrosophic hyper set, Single-valued Łukasiewicz Neutrosophic Super hyper set.

INTRODUCTION

Fuzzy set theory [11] handles inconsistent and ambiguous data by providing membership grades to the elements of the set stating its belongingness in the set, whereas Intuitionistic fuzzy set theory provides both the belongingness and non-belongingness of the element in a set. These set theories lack in handling uncertain data. To handle such type of data, [1] Smarandache proposed the concept of Neutrosophic Set, which discusses the elements based on three components namely, truth, indeterminacy and falsity in non-standard intervals. Also, it is not suitable for complicate problems due to the representation of non-standard intervals for the Neutrosophic components. This leads to the proposal of Single-valued Neutrosophic Set ([9], [10]), which maps the elements to Neutrosophic components under [0,1]. From the soft set theory, [6, 8] the author proposed the concept of Hyper Soft and Super Hyper soft set with some examples. These concepts help to work on mapping with multiple layers of membership values. Also, he stated the fuzzy extensions in hyper and super hyper soft sets [7]. The Łukasiewicz logic was proposed by Jan Łukasiewicz, a logician and Philosopher who provides a powerful framework with diverse application across various disciplines. ([2], [3], [4], [5]) Many authors worked this logic with fuzzy set theory in various algebraic structures.

In this chapter, we have proposed the neutrosophic concept in Hyper and Super Hyper set for mapping with multiple components of membership values. We have extended this concept in Łukasiewicz logic to prove mathematically, logical, consistent and bounded along with a discussion of some numerical example.

BACKGROUND

Definition 1. A Hyper set ($\mathcal{HS}$) $\hat{\mathcal{A}}$ over the universe $\hat{\mathcal{U}}$ is defined as

$$\hat{\mathcal{A}} = \left\{ \prec \breve{u}, \mu_{\hat{\mathcal{A}}}(\breve{u}) \succ : \breve{u} \in \hat{\mathcal{U}} \right\}$$

Where $\mu_{\hat{\mathcal{A}}}(\breve{u})$ is a set of membership values rather than a single membership value and it maps either to a binary value or in $[0,1]$ for each element $\breve{u}$ in $\hat{\mathcal{U}}$. $\mu_{\hat{\mathcal{A}}}(\breve{u})$ is called as hyper membership function for each element $\breve{u}$ in $\hat{\mathcal{U}}$.

Definition 2. A Super Hyper set ($\mathcal{SHS}$) $\hat{\mathcal{A}}$ over the universe $\hat{\mathcal{U}}$ is defined as

$$\hat{\mathcal{B}} = \left\{ \prec \breve{u}, M\mu_{\hat{\mathcal{A}}}(\breve{u}) \succ : \breve{u} \in \hat{\mathcal{U}} \right\}$$

Where $M\mu_{\hat{\mathcal{A}}}(\breve{u})$ is a set of sets of membership values rather than a single membership value or a set of membership values and it maps either to a binary value or in $[0,1]$ for each element $\breve{u}$ in $\hat{\mathcal{U}}$. $M\mu_{\hat{\mathcal{A}}}(\breve{u})$ is called as Super hyper membership function for each element $\breve{u}$ in $\hat{\mathcal{U}}$.

Definition 3. A Single valued Neutrosophic set ($\mathcal{SVNS}$) $\hat{\mathcal{C}}$ over the universe $\hat{\mathcal{U}}$ is of the form

$$\hat{\mathcal{C}} = \left\{ \prec \breve{u}, \hat{\mathfrak{t}}_{\hat{\mathcal{C}}}(\breve{u}), \hat{\mathfrak{i}}_{\hat{\mathcal{C}}}(\breve{u}), \hat{\mathfrak{f}}_{\hat{\mathcal{C}}}(\breve{u}) \succ : \breve{u} \in \hat{\mathcal{U}} \right\}$$

Where $\hat{\mathfrak{t}}_{\hat{\mathcal{C}}}(\breve{u}), \hat{\mathfrak{i}}_{\hat{\mathcal{C}}}(\breve{u})$ and $\hat{\mathfrak{f}}_{\hat{\mathcal{C}}}(\breve{u})$ are called truth, indeterminacy and falsity-membership functions, defined by

$$\hat{\mathfrak{t}}_{\hat{\mathcal{C}}}: \hat{\mathcal{U}} \to [0,1], \qquad \hat{\mathfrak{i}}_{\hat{\mathcal{C}}}: \hat{\mathcal{U}} \to [0,1], \qquad \hat{\mathfrak{f}}_{\hat{\mathcal{C}}}: \hat{\mathcal{U}} \to [0,1]$$

such that $0 \leq \hat{\mathfrak{t}}_{\hat{\mathcal{C}}}(\breve{u}) + \hat{\mathfrak{i}}_{\hat{\mathcal{C}}}(\breve{u}) + \hat{\mathfrak{f}}_{\hat{\mathcal{C}}}(\breve{u}) \leq 3$.

Example 4. Let $\hat{\mathcal{U}}$ be set of all Traffic Levels and $\hat{\mathcal{P}}$ be the set of factors affecting the Traffic.

$$\hat{\mathcal{U}} = \{\breve{u}_1 = Low\ Traffic, \breve{u}_2 = Moderate\ Traffic, \breve{u}_3 = Heavy\ Traffic\}$$

$$\hat{\mathcal{P}} = \{\breve{p}_1 = Time, \breve{p}_2 = Weather, \breve{p}_3 = Road\ accident\}$$

Now let us assign the set of truth, indeterminacy and falsity membership values for each traffic levels with respect to the factors affecting the traffic. The $\mathcal{SVNS}$ is constructed as follows

Time: $\breve{p}_1 = \left\{ \left(\breve{u}_1, (0.6,0.2,0.3) \right), \left(\breve{u}_2, (0.65,0.2,0.15) \right), \left(\breve{u}_3, (1,0,0) \right) \right\}$

Weather: $\breve{p}_2 = \left\{ \left(\breve{u}_1, (0.75,0.15,0.1) \right), \left(\breve{u}_2, (0.55,0.3,0.15) \right), \left(\breve{u}_3, (0.45,0.39,0.16) \right) \right\}$

Road Accident: $\breve{p}_3 = \left\{ \left(\breve{u}_1, (0.33,0.3,0.37) \right), \left(\breve{u}_2, (0.51,0.3,0.19) \right), \left(\breve{u}_3, (1,0,0) \right) \right\}$

Here, "Low traffic" with respect to "Time" maps to the neutrosophic component $(0.6,0.2,0.3)$. This represents that 60% is the truth that time of a day cause low traffic, 20% is uncertainty or lack of information that time of a day cause low traffic and 30% is the falsity or low probability that time of a day cause low traffic. Similarly, we can say it for other cases.

Definition 5. A Single valued Neutrosophic Hyper set ($\mathcal{SVNHS}$) $\hat{\mathcal{D}}$ over the universe $\hat{\mathcal{U}}$ is defined by

$$\hat{\mathcal{D}} = \left\{ \prec \breve{u}, \hat{\mathcal{T}}_{\hat{\mathcal{D}}}(\breve{u}), \hat{\mathcal{I}}_{\hat{\mathcal{D}}}(\breve{u}), \hat{\mathcal{F}}_{\hat{\mathcal{D}}}(\breve{u}) \succ : \breve{u} \in \hat{\mathcal{U}} \right\}$$

Where $\hat{\mathcal{T}}_{\hat{\mathcal{D}}}: \hat{\mathcal{U}} \to \mathfrak{P}[0,1], \hat{\mathcal{I}}_{\hat{\mathcal{D}}}: \hat{\mathcal{U}} \to \mathfrak{P}[0,1]$ and $\hat{\mathcal{F}}_{\hat{\mathcal{D}}}: \hat{\mathcal{U}} \to \mathfrak{P}[0,1]$ are set of truth, indeterminacy and falsity-membership values, associated with the element $\breve{u}$ in $\hat{\mathcal{U}}$.

Example 6. Let us discuss the same example for $\mathcal{SVNHS}$.

The $\mathcal{SVNHS}$ offers each element in the universe of discourse to map with more than one neutrosophic component. This can be constructed as follows

Time: $\breve{\mathcal{P}}_1 = \begin{Bmatrix} (\breve{u}_1, \{(0.4,0.3,0.3), (0.6,0.2,0.3)\}), (\breve{u}_2, \{(0.6,0.2,0.3), (0.65,0.2,0.15)\}), \\ (\breve{u}_3, \{(0.8,0.09,0.11), (1,0,0)\}) \end{Bmatrix}$

Weather: $\breve{\mathcal{P}}_2 = \begin{Bmatrix} (\breve{u}_1, \{(0.63,0.19,0.18), (0.75,0.15,0.1)\}), (\breve{u}_2, \{(0.4,0.3,0.3), (0.55,0.3,0.15)\}), \\ (\breve{u}_3, \{(0.15,0.82,0.03), (0.45,0.39,0.16)\}) \end{Bmatrix}$

Road Accident: $\breve{\mathcal{P}}_3 = \begin{Bmatrix} (\breve{u}_1, \{(0.2,0.4,0.4), (0.33,0.3,0.37)\}), (\breve{u}_2, \{(0.32,0.5,0.18), (0.51,0.3,0.19)\}), \\ (\breve{u}_3, \{(0.55,0.35,0.1), (1,0,0)\}) \end{Bmatrix}$

Here, "Low traffic" with respect to "Time" maps to the set $\{(0.4,0.3,0.3), (0.6,0.2,0.3)\}$ with elements as of the form of Neutrosophic components. This represents that it maps either $(0.4,0.3,0.3)$ or $(0.6,0.2,0.3)$. This says that 40% or 60% is the truth that time of a day cause low traffic, 30% or 20% is uncertainty or lack of information that time of a day cause low traffic and 30% is the falsity or low probability that time of a day cause low traffic. Similarly, we can say it for other cases.

Definition 7. A Single-valued Neutrosophic Super Hyper Set ($\mathcal{SVNSHS}$) $\widehat{\mathcal{GH}}$ over the universe $\hat{\mathcal{U}}$ is defined by

$$\widehat{\mathcal{GH}} = \left\{ \prec \breve{u}, \hat{\mathcal{T}}_{\widehat{\mathcal{GH}}}(\breve{u}), \hat{\mathcal{I}}_{\widehat{\mathcal{GH}}}(\breve{u}), \hat{\mathcal{F}}_{\widehat{\mathcal{GH}}}(\breve{u}) \succ : \breve{u} \in \hat{\mathcal{U}} \right\}$$

Where $\hat{\mathcal{T}}_{\widehat{\mathcal{GH}}}: \hat{\mathcal{U}} \to \mathfrak{P}[\mathfrak{P}[0,1]], \hat{\mathcal{I}}_{\widehat{\mathcal{GH}}}: \hat{\mathcal{U}} \to \mathfrak{P}[\mathfrak{P}[0,1]]$ and $\hat{\mathcal{F}}_{\widehat{\mathcal{GH}}}: \hat{\mathcal{U}} \to \mathfrak{P}[\mathfrak{P}[0,1]]$ are set of sets of truth, indeterminacy and falsity-membership values, associated with the element $\breve{u}$ in $\hat{\mathcal{U}}$.

Example 8. Now, let us discuss the same example for $\mathcal{SVNHS}$

Here, we subdivide each traffic levels in multiple factors. As a part of it $\breve{\mathcal{P}}_1, \breve{\mathcal{P}}_2$ and $\breve{\mathcal{P}}_3$ are also sets with detailed factors as

$$\breve{\mathcal{P}}_1 = \{\breve{\breve{\mathcal{P}}}\breve{t}_1 = Morning\ time, \breve{\breve{\mathcal{P}}}\breve{t}_2 = Night\ time, \breve{\breve{\mathcal{P}}}\breve{t}_3 = Evening\ time, \breve{\breve{\mathcal{P}}}\breve{t}_4 = Peak\ hours\ \}$$

$$\breve{\mathcal{P}}_2 = \{\breve{\mathcal{P}}\breve{w}_1 = Clear, \breve{\mathcal{P}}\breve{w}_2 = Rainy, \breve{\mathcal{P}}\breve{w}_3 = Cloudy, \breve{\mathcal{P}}\breve{w}_4 = Drizzle, \breve{\mathcal{P}}\breve{w}_5 = Storm, \breve{\mathcal{P}}\breve{w}_6 = Heavy\ rain\}$$

$$\breve{\mathcal{P}}_3 = \{\breve{\mathcal{P}}\breve{r}_1 = Minor\ accident, \breve{\mathcal{P}}\breve{r}_2 = Major\ accident, \breve{\mathcal{P}}\breve{r}_3 = Few\ accident, \breve{\mathcal{P}}\breve{r}_4 = Moderate\ accident,$$
$$\breve{\mathcal{P}}\breve{r}_5 = Severe\ accident, \breve{\mathcal{P}}\breve{r}_6 = No\ accident\}$$

Now let us assign the set of truth, indeterminacy and falsity membership values for each traffic levels with respect to the set of factors affecting the traffic. The $\mathcal{SVNSHS}$ is constructed as follows

$$\text{Time: } \breve{\mathscr{P}}_1 = \begin{pmatrix} \left(\breve{u}_1, \left\{\left(\breve{\mathscr{P}}t_1, (0.4,0.3,0.3)\right), \left(\breve{\mathscr{P}}t_2, (0.6,0.2,0.3)\right)\right\}\right), \\ \left(\breve{u}_2, \left\{\left(\breve{\mathscr{P}}t_1, (0.6,0.2,0.3)\right), \left(\breve{\mathscr{P}}t_3, (0.65,0.2,0.15)\right)\right\}\right), \\ \left(\breve{u}_3, \left\{\left(\breve{\mathscr{P}}t_4, (0.8,0.09,0.11)\right), \left(\breve{\mathscr{P}}t_2, (1,0,0)\right)\right\}\right) \end{pmatrix}$$

$$\text{Weather: } \breve{\mathscr{P}}_2 = \begin{pmatrix} \left(\breve{u}_1, \left\{\left(\breve{\mathscr{P}}w_1, (0.63,0.19,0.18)\right), \left(\breve{\mathscr{P}}w_2, (0.75,0.15,0.1)\right)\right\}\right), \\ \left(\breve{u}_2, \left\{\left(\breve{\mathscr{P}}w_3, (0.4,0.3,0.3)\right), \left(\breve{\mathscr{P}}w_4, (0.55,0.3,0.15)\right)\right\}\right), \\ \left(\left(\breve{u}_3, \left\{\left(\breve{\mathscr{P}}w_5, (0.15,0.82,0.03)\right), \left(\breve{\mathscr{P}}w_6, (0.45,0.39,0.16)\right)\right\}\right)\right) \end{pmatrix}$$

$$\text{Road Accident: } \breve{\mathscr{P}}_3 = \begin{pmatrix} \left(\breve{u}_1, \left\{\left(\breve{\mathscr{P}}r_1, (0.2,0.4,0.4)\right), \left(\breve{\mathscr{P}}r_6, (0.33,0.3,0.37)\right)\right\}\right), \\ \left(\breve{u}_2, \left\{\left(\breve{\mathscr{P}}r_4, (0.32,0.5,0.18)\right), \left(\breve{\mathscr{P}}r_3, (0.51,0.3,0.19)\right)\right\}\right), \\ \left(\breve{u}_3, \left\{\left(\breve{\mathscr{P}}r_5, (0.55,0.35,0.1)\right), \left(\breve{\mathscr{P}}r_2, (1,0,0)\right)\right\}\right) \end{pmatrix}$$

Here, "Low traffic" with respect to "Time" maps to morning time and night time with neutrosophic components as $(0.4,0.3,0.3)$ and $(0.6,0.2,0.3)$ respectively. This says that Low traffic in morning time has 40% truth, 30% indeterminacy and 30% falsity. The Low traffic in Night time has 60% truth, 20% indeterminacy and 30% falsity. Similarly, we can say it for other cases.

SINGLE VALUED ŁUKASIEWICZ NEUTROSOPHIC SUPER HYPER SET

Definition 9. Let $\hat{\mathcal{C}}$ be a $\mathcal{SVNS}$ in $\hat{\mathcal{U}}$ and let $\ddot{\theta} \in [0,1]$. A single valued Łukasiewicz Neutrosophic set ($\mathcal{SVLNS}$) $\hat{\mathcal{L}}_{\hat{\mathcal{C}}}^{\ddot{\theta}}$ is defined as

$$\hat{\mathcal{L}}_{\hat{\mathcal{C}}}^{\ddot{\theta}} = \left\{ \prec \breve{u}, \hat{\mathfrak{t}}_{\hat{\mathcal{L}}_{\hat{\mathcal{C}}}^{\ddot{\theta}}}(\breve{u}), \hat{\mathfrak{i}}_{\hat{\mathcal{L}}_{\hat{\mathcal{C}}}^{\ddot{\theta}}}(\breve{u}), \hat{\mathfrak{f}}_{\hat{\mathcal{L}}_{\hat{\mathcal{C}}}^{\ddot{\theta}}}(\breve{u}) \succ : \breve{u} \in \hat{\mathcal{U}} \right\}$$

Where $\hat{\mathfrak{t}}_{\hat{\mathcal{L}}_{\hat{\mathcal{C}}}^{\ddot{\theta}}}: \hat{\mathcal{U}} \to [0,1], \breve{u} \to max\{0, \hat{\mathfrak{t}}_{\hat{\mathcal{C}}}(\breve{u}) + \ddot{\theta} - 1\}$, $\hat{\mathfrak{i}}_{\hat{\mathcal{L}}_{\hat{\mathcal{C}}}^{\ddot{\theta}}}: \hat{\mathcal{U}} \to [0,1], \breve{u} \to max\{0, \hat{\mathfrak{i}}_{\hat{\mathcal{C}}}(\breve{u}) + \ddot{\theta} - 1\}$ and $\hat{\mathfrak{f}}_{\hat{\mathcal{L}}_{\hat{\mathcal{C}}}^{\ddot{\theta}}}: \hat{\mathcal{U}} \to [0,1], \breve{u} \to max\{0, \hat{\mathfrak{f}}_{\hat{\mathcal{C}}}(\breve{u}) + \ddot{\theta} - 1\}$ are truth, indeterminacy and falsity membership functions respectively.

Such that $0 \leq \hat{\mathfrak{t}}_{\hat{\mathcal{L}}_{\hat{\mathcal{C}}}^{\ddot{\theta}}}(\breve{u}) + \hat{\mathfrak{i}}_{\hat{\mathcal{L}}_{\hat{\mathcal{C}}}^{\ddot{\theta}}}(\breve{u}) + \hat{\mathfrak{f}}_{\hat{\mathcal{L}}_{\hat{\mathcal{C}}}^{\ddot{\theta}}}(\breve{u}) \leq 3$.

Example 10. Let us consider the Example 4 and convert it to $\mathcal{SVLNS}$ by taking $\ddot{\theta} = 0.65$ in the definition.

Thus, the $\mathcal{SVLNS}$ is constructed as below

Time: $\breve{\mathscr{P}}_1 = \left\{\left(\breve{u}_1, (0.25,0,0)\right), \left(\breve{u}_2, (0.3,0,0)\right), \left(\breve{u}_3, \{(0.65,0,0)\}\right)\right\}$

Weather: $\breve{\mathscr{P}}_2 = \left\{\left(\breve{u}_1, (0.4,0,0)\right), \left(\breve{u}_2, (0.2,0,0)\right), \left(\breve{u}_3, (0.1,0.04,0)\right)\right\}$

Road Accident: $\breve{\mathscr{P}}_3 = \left\{\left(\breve{u}_1, (0,0,0)\right), \left(\breve{u}_2, (0.16,0,0)\right), \left(\breve{u}_3, (0.65,0,0)\right)\right\}$

Here, "Low traffic" with respect to "Time" maps to the neutrosophic component $(0.25,0,0)$. This represents that 25% is the truth that time of a day cause low traffic and there is 0% lack of information and falsity in time of a day cause low traffic. Thus, it helps to eliminate or reduce the indeterminacy and falsity occurs in the data. Similarly, we can say it for other cases.

Definition 11. Let $\widehat{\mathcal{D}}$ be a $\mathcal{SVNHS}$ in $\hat{\mathcal{U}}$ and let $\ddot{\theta} \in [0,1]$. A single valued Łukasiewicz Neutrosophic Hyper set ($\mathcal{SVLNHS}$) $\hat{\mathcal{L}}_{\widehat{\mathcal{D}}}^{\ddot{\theta}}$ is defined as

$$\hat{\mathcal{L}}_{\hat{\mathcal{D}}}^{\ddot{\theta}} = \left\{ \prec \breve{u}, \acute{\mathcal{T}}_{\hat{\mathcal{L}}_{\hat{\mathcal{D}}}^{\ddot{\theta}}}(\breve{u}), \acute{\mathcal{I}}_{\hat{\mathcal{L}}_{\hat{\mathcal{D}}}^{\ddot{\theta}}}(\breve{u}), \acute{\mathcal{F}}_{\hat{\mathcal{L}}_{\hat{\mathcal{D}}}^{\ddot{\theta}}}(\breve{u}) \succ : \breve{u} \in \hat{\mathcal{U}} \right\}$$

Where $\acute{\mathcal{T}}_{\hat{\mathcal{L}}_{\hat{\mathcal{D}}}^{\ddot{\theta}}} : \hat{\mathcal{U}} \to \mathfrak{P}([0,1]), \breve{u} \to max\{0, \acute{\mathcal{T}}_{\hat{\mathcal{D}}}(\breve{u}) + \ddot{\theta} - 1\}$, $\acute{\mathcal{I}}_{\hat{\mathcal{L}}_{\hat{\mathcal{D}}}^{\ddot{\theta}}}(\breve{u}) : \hat{\mathcal{U}} \to \mathfrak{P}([0,1]), \breve{u} \to max\{0, \acute{\mathcal{I}}_{\hat{\mathcal{D}}}(\breve{u}) + \ddot{\theta} - 1\}$ and $\acute{\mathcal{F}}_{\hat{\mathcal{L}}_{\hat{\mathcal{D}}}^{\ddot{\theta}}}(\breve{u}) : \hat{\mathcal{U}} \to \mathfrak{P}([0,1]), \breve{u} \to max\{0, \acute{\mathcal{F}}_{\hat{\mathcal{D}}}(\breve{u}) + \ddot{\theta} - 1\}$ are set of truth, indeterminacy and falsity-membership values, associated with the element $\breve{u}$ in $\hat{\mathcal{U}}$. These sets are contained in [0,1]. Here $\acute{\mathcal{T}}_{\hat{\mathcal{D}}}(\breve{u}) + \ddot{\theta} = \left\{ \breve{\nu} + \ddot{\theta} | \breve{\nu} \in \acute{\mathcal{T}}_{\hat{\mathcal{D}}}(\breve{u}) \right\}$, $\acute{\mathcal{I}}_{\hat{\mathcal{D}}}(\breve{u}) + \ddot{\theta} = \left\{ \breve{\nu} + \ddot{\theta} | \breve{\nu} \in \acute{\mathcal{I}}_{\hat{\mathcal{D}}}(\breve{u}) \right\}$ and $\acute{\mathcal{F}}_{\hat{\mathcal{D}}}(\breve{u}) + \ddot{\theta} = \left\{ \breve{\nu} + \ddot{\theta} | \breve{\nu} \in \acute{\mathcal{F}}_{\hat{\mathcal{D}}}(\breve{u}) \right\}$.

Example 12. Let us consider the Example 6 and convert it to $\mathcal{SVLNHS}$ by taking $\ddot{\theta} = 0.65$ in the definition.

Thus, the $\mathcal{SVLNHS}$ is constructed as below

$$\text{Time: } \breve{\mathcal{P}}_1 = \left\{ \begin{matrix} (\breve{u}_1, \{(0.05,0,0), (0.25,0,0)\}), (\breve{u}_2, \{(0.25,0,0), (0.3,0,0)\}), \\ (\breve{u}_3, \{(0.45,0,0.11), (0.65,0,0)\}) \end{matrix} \right\}$$

$$\text{Weather: } \breve{\mathcal{P}}_2 = \left\{ \begin{matrix} (\breve{u}_1, \{(0.28,0,0), (0.4,0,0)\}), (\breve{u}_2, \{(0.05,0,0), (0.2,0,0)\}), \\ (\breve{u}_3, \{(0,0.47,0), (0.1,0.04,0)\}) \end{matrix} \right\}$$

$$\text{Road Accident: } \breve{\mathcal{P}}_3 = \left\{ \begin{matrix} (\breve{u}_1, \{(0,0.05,0.05), (0,0,0)\}), (\breve{u}_2, \{(0,0.15,0), (0.16,0,0)\}), \\ (\breve{u}_3, \{(0.2,0,0), (0.65,0,0)\}) \end{matrix} \right\}$$

Here, "Low traffic" with respect to "Time" maps to the set $\{(0.05,0,0), (0.25,0,0)\}$ with elements as of the form of Neutrosophic components. This represents that it maps either $(0.05,0,0)$ or $(0.25,0,0)$. This says that 5% or 25% is the truth that time of a day cause low traffic and there is 0% lack of information and falsity in time of a day cause low traffic. Thus, it helps to map with more than one neutrosophic component along with the reduction and elimination of indetermined and false data values. Similarly, we can say it for other cases.

Definition 13. Let $\widehat{\mathcal{GH}}$ be a $\mathcal{SVNSHS}$ in $\hat{\mathcal{U}}$ and let $\ddot{\theta} \in [0,1]$. A Single-valued Łukasiewicz Neutrosophic Super Hyper Set ($\mathcal{SVLNSHS}$) $\hat{\mathcal{L}}_{\widehat{\mathcal{GH}}}^{\ddot{\theta}}$ is defined as

$$\hat{\mathcal{L}}_{\widehat{\mathcal{GH}}}^{\ddot{\theta}} = \left\{ \prec \breve{u}, \acute{\mathcal{T}}_{\hat{\mathcal{L}}_{\widehat{\mathcal{GH}}}^{\ddot{\theta}}}(\breve{u}), \acute{\mathcal{I}}_{\hat{\mathcal{L}}_{\widehat{\mathcal{GH}}}^{\ddot{\theta}}}(\breve{u}), \acute{\mathcal{F}}_{\hat{\mathcal{L}}_{\widehat{\mathcal{GH}}}^{\ddot{\theta}}}(\breve{u}) \succ : \breve{u} \in \hat{\mathcal{U}} \right\}$$

Where $\acute{\mathcal{T}}_{\hat{\mathcal{L}}_{\widehat{\mathcal{GH}}}^{\ddot{\theta}}} : \hat{\mathcal{U}} \to \mathfrak{P}(\mathfrak{P}([0,1])), \breve{u} \to max\{0, \acute{\mathcal{T}}_{\widehat{\mathcal{GH}}}(\breve{u}) + \ddot{\theta} - 1\}$, $\acute{\mathcal{I}}_{\hat{\mathcal{L}}_{\widehat{\mathcal{GH}}}^{\ddot{\theta}}} : \hat{\mathcal{U}} \to \mathfrak{P}(\mathfrak{P}([0,1])), \breve{u} \to max\{0, \acute{\mathcal{I}}_{\widehat{\mathcal{GH}}}(\breve{u}) + \ddot{\theta} - 1\}$, and $\acute{\mathcal{F}}_{\hat{\mathcal{L}}_{\widehat{\mathcal{GH}}}^{\ddot{\theta}}} : \hat{\mathcal{U}} \to \mathfrak{P}(\mathfrak{P}([0,1])), \breve{u} \to max\{0, \acute{\mathcal{F}}_{\widehat{\mathcal{GH}}}(\breve{u}) + \ddot{\theta} - 1\}$, are set of sets of truth, indeterminacy and falsity-membership values, associated with the element $\breve{u}$ in $\hat{\mathcal{U}}$. These sets are contained in [0,1]. Here $\acute{\mathcal{T}}_{\widehat{\mathcal{GH}}}(\breve{u}) + \ddot{\theta} = \left\{ \breve{\nu} + \ddot{\theta} | \breve{\nu} \in \acute{\mathcal{T}}_{\widehat{\mathcal{GH}}}(\breve{u}) \right\}$, $\acute{\mathcal{I}}_{\widehat{\mathcal{GH}}}(\breve{u}) + \ddot{\theta} = \left\{ \breve{\nu} + \ddot{\theta} | \breve{\nu} \in \acute{\mathcal{I}}_{\widehat{\mathcal{GH}}}(\breve{u}) \right\}$ and $\acute{\mathcal{F}}_{\widehat{\mathcal{GH}}}(\breve{u}) + \ddot{\theta} = \left\{ \breve{\nu} + \ddot{\theta} | \breve{\nu} \in \acute{\mathcal{F}}_{\widehat{\mathcal{GH}}}(\breve{u}) \right\}$.

Example 14. Let us consider the Example 8 and convert it to $\mathcal{SVLNSHS}$ by taking $\ddot{\theta} = 0.65$ in the definition. Thus, the $\mathcal{SVLNSHS}$ is constructed as below

$$\text{Time: } \breve{\mathcal{P}}_1 = \left\{ \begin{matrix} \left(\breve{u}_1, \left\{ \left(\breve{\breve{p}}\breve{t}_1, (0.05,0,0) \right), \left(\breve{\breve{p}}\breve{t}_2, (0.25,0,0) \right) \right\}\right), \\ \left(\breve{u}_2, \left\{ \left(\breve{\breve{p}}\breve{t}_1, (0.25,0,0) \right), \left(\breve{\breve{p}}\breve{t}_3, (0.3,0,0) \right) \right\}\right), \\ \left(\breve{u}_3, \left\{ \left(\breve{\breve{p}}\breve{t}_4, (0.45,0,0.11) \right), \left(\breve{\breve{p}}\breve{t}_2, (0.65,0,0) \right) \right\}\right) \end{matrix} \right\}$$

$$\text{Weather: } \breve{\mathcal{P}}_2 = \left\{ \begin{matrix} \left(\breve{u}_1, \left\{ \left(\breve{\breve{p}}\breve{w}_1, (0.28,0,0) \right), \left(\breve{\breve{p}}\breve{w}_2, (0.4,0,0) \right) \right\}\right), \\ \left(\breve{u}_2, \left\{ \left(\breve{\breve{p}}\breve{w}_3, (0.05,0,0) \right), \left(\breve{\breve{p}}\breve{w}_4, (0.2,0,0) \right) \right\}\right), \\ \left(\breve{u}_3, \left\{ \left(\breve{\breve{p}}\breve{w}_5, (0,0.47,0) \right), \left(\breve{\breve{p}}\breve{w}_6, (0.1,0.04,0) \right) \right\}\right) \end{matrix} \right\}$$

$$\text{Road Accident: } \breve{\mathcal{P}}_3 = \begin{pmatrix} \left(\breve{u}_1, \left\{\left(\breve{\mathcal{P}}\breve{r}_1, (0,0.05,0.05)\right)\right\}\right), \\ \left(\breve{u}_2, \left\{\left(\breve{\mathcal{P}}\breve{r}_4, (0,0.15,0)\right), \left(\breve{\mathcal{P}}\breve{r}_3, (0.16,0,0)\right)\right\}\right), \\ \left(\breve{u}_3, \left\{\left(\breve{\mathcal{P}}\breve{r}_5, (0.2,0,0)\right), \left(\breve{\mathcal{P}}\breve{r}_2, (0.65,0,0)\right)\right\}\right) \end{pmatrix}$$

Here, “Low traffic” with respect to “Time” maps to morning time and night time with neutrosophic components as $(0.4,0.3,0.3)$ and $(0.6,0.2,0.3)$ respectively. This says that Low traffic in morning time has 40% truth, 30% indeterminacy and 30% falsity. The Low traffic in Night time has 60% truth, 20% indeterminacy and 30% falsity. Similarly, we can say it for other cases.

CONCLUSION

In this chapter, we have inquired over the concepts of single-valued neutrosophic hyper and super hyper sets. This allows, one to construct a complex system with elements having multiple layers of dependencies. We have incorporated them into the theory of Łukasiewicz to enhance the reliability of data analysis by facilitating the drop down or removal of indeterminate and false values. This approach offers a robust mathematical basis for scenarios that require reliable truth evaluations. In future, this framework can be extended in machine learning models for the improved results in medical diagnosis and decision-making problem.

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Chapter 14

Nano Topology via Hypergroup

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Abstract

This chapter deals with hypergroup theory in nano topology. We also introduce a new class of nano open sets which form a nano topology. Moreover, we show that from hypergroup we can form nano topology and vice versa. Finally, we obtain invertible subhypergroup nano topology which is coarser than nano topology through least open sets.

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1 Introduction

"Hyperstructures were introduced in 1934 by a French mathematician,in 8th Congress of Scandinavian Mathematician[1]. Pawlak introduced rough set theory[11] for dealing with vagueness or uncertainity. Nowadays hyperstructures are studied for their applications in pure and applied mathematics for example,semihypergroups are the simplest algebraic hyperstructures that exhibits the properties of closure and associativity. The theory and applications of rough sets have impressively developed in the year 1982. Algebraic hyperstructures have been studied both from the theoretical point of view and especially for their applications in other fields such as non Euclidean geometries, hypergraphs, fuzzy sets,cryptography, artificial intelligence, codes. Nano topology is studied by Lellis Thivagar and Carmel Richard[9]. In this chapter, we present a general framework for the study of approximations in fundamental relation and study about least open set which forms nano topology. Further, we obtain the result which indicates that invertible subhypergroup nano topology is coarser than least open set nano topology. Finally, we apply the above approach in nano topology via hypergroup".

2 Preliminaries

In this section we present the some basic notations and results about hypergroups which will be necessary in sequel.

Definition 2.1 *[2]:"A hyperoperation on H is a map* $\circ : H \times H \longrightarrow P^*(H)$*, where H is a nonempty set and* $P^*(H)$ *is the set of all nonempty subsets of H and for any nonempty subsets A,B of H where* $A \circ B = \cup_{a \in A, b \in B} a \circ b$*. The couple (H,* $\circ$*) is called a hypergroupoid ".*

Definition 2.2 *[2]:A hypergroupoid (H,*$\circ$*) is a semihypergroup if* $\forall$ *x,y,z of H we have (xy)z = x(yz).*

Definition 2.3 *[2]: Let (H,* $\circ$*) is a hypergroup if* $\forall$ *x,y,z of H, we have (xy)z = x(yz) and Hx = xH = H.*

Definition 2.4 *[2]: "Let* $K \subset H$*,* $K \neq \emptyset$ *and (K,* $\circ$*) be a subhypergroup of (H,* $\circ$*). If for any a* $\in K$*, then we have Ka = aK = K. A subhypergroup K of H is*

(i) closed if $\forall$ *a,b of K and x,y of* $H \ni a \in bx$ *and* $a \in yb \Rightarrow$ *x,y belong to K.*

(ii) left invertible if $\forall$ *x,y of* $H \ni x \in Ky \Rightarrow y \in Kx$*. We say that K is invertible if it is left and right invertible.*

(iii) normal if for any $x \in H$*, then xK = Kx."*

Proposition 2.5 *[2]:(i) K is right invertible in* $H \Longleftrightarrow$ *the following condition holds:* $b \in Ka \Longrightarrow a \in Kb \ \forall$ *a,b* $\in H$*.*
(ii) K is left invertible in $H \Longleftrightarrow$ *the following condition holds:* $b \in aK \Longrightarrow a \in bK \ \forall$ *a,b* $\in H$*.*

Proposition 2.6 *[2]:(i) K is right invertible in* $H \Longleftrightarrow$ *the following condition is valid :* $Ka \neq Kb \Longrightarrow Ka \cap Kb = \emptyset \ \forall$ *a,b* $\in H$*.*
(ii) K is left invertible in $H \Longleftrightarrow$ *the following condition is valid:* $aK \neq bK \Longrightarrow aK \cap bK = \emptyset$ $\forall$ *a,b* $\in H$*.*

Definition 2.7 *:"Let (H,* $\circ$*) be a hypergroup, S be an invertible subhypergroup of H.* $\forall$ $x \in H$*, xS induces an equivalence relation on H and* $A \subseteq H$*, where* $L_S(A)$*,* $U_S(A)$*,* $B_S(A)$ *are follows.*

(i) The lower approximation of A with respect to S is the set of all objects, which is classified as A with respect to S and it is denoted by $L_S(A)$*,* $L_S(A) = \{x \in H | xS \subseteq A\}$*.*

(ii) The upper approximation of A with respect to S,is the set of all objects, which is classified as A with respect to S and it is denoted by $U_S(A)$*,* $U_S(A) = \{x \in H | xS \cap A \neq \emptyset\}$*.*

(iii) The boundary region of A with respect to S, is the set of all objects, which can be classified as A with respect to S and it is denoted by $B_S(A)$*,* $B_S(A) = U_S(A) - L_S(A)$*."*

Definition 2.8 *[9]:Let* $\mathcal{U}$ *be an universe, R be an equivalence relation on* $\mathcal{U}$ *and* $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X), U_R(X), B_R(X)$ *and* $X \subseteq \mathcal{U}$*,* $\tau_R(X)$ *satisfies the following axioms:*

(i) $\mathcal{U}$ *and* $\emptyset$ *belongs to* $\tau_R(X)$*.*

(ii) The arbitrary union of the elements in $\tau_R(X)$ *is again in* $\tau_R(X)$*.*

(iii) The finite intersection of elements in $\tau_R(X)$ *is in* $\tau_R(X)$*.*

where $\tau_R(X)$ *forms a topology on* $\mathcal{U}$ *called the nano topology (NT) on* $\mathcal{U}$ *with respect to X and (*$\mathcal{U}$*,*$\tau_R(X)$*) is known as nano topological space(NTS). The elements of* $\tau_R(X)$ *are called nano-open sets.*

Proposition 2.9 *[2]:"Let* $\mathcal{U}$ *be an finite universe and* $X \subset \mathcal{U}$*,* $\mathcal{U}/R$ *be an indiscernibility relation on* $\mathcal{U}$ *then*

(i) ***Nano Type-1***$(\mathcal{NT}_1)$: *The nano topology* $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X)\}$ *when* $L_R(X) = U_R(X) = X$.

(ii) ***Nano Type-2***$(\mathcal{NT}_2)$:*The nano topology* $\tau_R(X) = \{\mathcal{U}, \emptyset, U_R(X)\}$ *when* $L_R(X) = \emptyset$ *and* $U_R(X) \neq \mathcal{U}$.

(iii) ***Nano Type-3***$(\mathcal{NT}_3)$: *The nano topology* $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X), B_R(X)\}$ *when* $L_R(X) \neq \emptyset$ *and* $U_R(X) = \mathcal{U}$

(iv) ***Nano Type-4***$(\mathcal{NT}_4)$: *The nano topology* $\tau_R(X) = \{\mathcal{U}, \emptyset\}$ *when* $L_R(X) = \emptyset$ *and* $U_R(X) = \mathcal{U}$ *is the indiscrete nano topology on* $\mathcal{U}$.

(v) ***Nano Type-5***$(\mathcal{NT}_5)$: *The nano topology* $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X), U_R(X), B_R(X)\}$ *when* $U_R(X) \neq \mathcal{U}$, $L_R(X) \neq U_R(X)$, $L_R(X) \neq \emptyset$."

Definition 2.10 *:Let* $(\mathcal{U},\tau_R(X))$ *be a nano topological space on* $\mathcal{U}$ *with respect to* X *and a subset* $\mathcal{Y}$ *on* $\mathcal{U}$ *such that* $X \subseteq \mathcal{Y} \subseteq \mathcal{U}$ *and* $(\mathcal{Y}/R) \subseteq \mathcal{U}/R$ *then the relative nano topology on* $\mathcal{Y}$ *is defined by* $\tau_R^*(X) = \{\mathcal{U},\emptyset, L_R^*(X), U_R^*(X), B_R^*(X)\}$
where, $L_R^*(X) = \{ L_R(X) \cap \mathcal{Y} / L_R(X) \cap \mathcal{Y} \subseteq X, L_R(X) \in \tau_R(X)\}$
$U_R^*(X) = \{ U_R(X) \cap \mathcal{Y} / U_R(X) \cap \mathcal{Y} \neq \emptyset, U_R(X) \in \tau_R(X)\}$
$B_R^*(X) = U_R^*(X) - L_R^*(X)$.

Definition 2.11 *[9]:Let* $(\mathcal{U},\tau_R(X))$ *and* $(\mathcal{V},\tau_{R_1}(Y)$ *be two nano topological space then a mapping* $f:\mathcal{U} \longrightarrow \mathcal{V}$ *is said to be nano continuous on* $\mathcal{U}$ *if the inverse image of every nano open set in* $\mathcal{V}$ *is nano open in* $\mathcal{U}$.

Definition 2.12 *: Let* $\mathcal{H}$ *be a hypergroup, S is an invertible subhypergroup(IS) on H and* $\tau_S(A) = \{\mathcal{H}, \emptyset, L_S(A), U_S(A), B_S(A)\}$ *where* $A \subseteq H$. $\tau_S(A)$ *satisfies the following axioms:*

(i) $\mathcal{H}$ *and* $\emptyset \in \tau_S(A)$.

(ii) *The union of the elements of any subcollection of* $\tau_S(A)$ *is in* $\tau_S(A)$.

(iii) *The intersection of the elements of any finite subcollection of* $\tau_S(A)$ *is in* $\tau_S(A)$.

That is $\tau_S(A)$ *forms a ISNT on* $\mathcal{H}$ *with respect to A. We call* $(\mathcal{H}$ *,*$\tau_S(A))$ *as the ISNTS. The elements are called IS nano open sets.*

Remark 2.13 *:We have* $L_S(X) = L_{R_S}(X)$ *and* $U_S(X) = U_{R_S}(X)$ *and* $B_S(X) = B_{R_S}(X)$*,where* R_S *is the equivalence relation with respect to IS. Indeed for any* $x \in \mathcal{H}$ *, the equivalence class of x with respect to* R_S *is xS.*

3 Nano topology via smallest equivalence relation

In this section we discuss the smallest equivalence relation through nano topology.

Proposition 3.1 *:"Let* $\mathcal{H}$ *be a semihypergroup(SH). The relation* β^* *is the smallest equivalence relation on* $\mathcal{H} \ni$ *the quotient* $\mathcal{H}$ */*β^**, the set of all equivalence classes is a semigroup.* β^* *is known as the fundamental equivalence relation on* $\mathcal{H}$ *which is studied by Corsini[2] concerning hypergroups. If U denotes the set of all the finite products of elements of* $\mathcal{H}$ *, then the relation* β *can be defined on* $\mathcal{H}$*, whose equivalence class is the fundamental relation* β^**[4]. For relation* β*, x and y in* $\mathcal{H}$ *we write* $x\beta y \iff \{x,y\} \in u$ *for some* $u \in U$. *We can rewrite the definition of* β^* *on* $\mathcal{H}$ *as follows:* $a\beta^* b \iff$ *there exist* $z_1, z_2, ...z_{n+1} \in H$ *with* $z_1 = a$, $z_{n+1} = b$ *and* $u_1, ...u_n \in U$ *such that* $\{z_i, z_{i+1}\} \in u_i (i=1,.....n)$. *Suppose* $\beta^*(a)$ *is the equivalence class containing* $a \in \mathcal{H}$. *Then the product* $\circ$ *on* $\mathcal{H}$ */*β^* *is defined as follows :* $\beta^*(a) \circ \beta^*(b) = \{\beta^*(c) | c \in \beta^*(a) \circ \beta^*(b)\}$ *for all a,b* $\in H$."

Definition 3.2 *:Let $\mathcal{H}$ be a SH. For a subset $X \subseteq \mathcal{H}$ we define the approximations of X relative to the fundamental equivalence relation β^* as follows:*

(i) $L_{\beta^*}(X) = \{x \in \mathcal{H} | \beta^*(x) \subseteq X\}$

(ii) $U_{\beta^*}(X) = \{x \in \mathcal{H} | \beta^*(x) \cap X \neq \emptyset\}$

(iii) $B_{\beta^*}(X) = U_{\beta^*}(X) - L_{\beta^*}(X)$

Definition 3.3 *: Let $\mathcal{H}$ = {p,q,r} and $\mathcal{H}/\beta^*$ = {{p},{q},{r}} where $\beta^*(p)$ = {p},$\beta^*(q)$ = {q},$\beta^*(r)$ = {r}. Now $\beta^*(p) \circ \beta^*(p)$ = {p},$\beta^*(q) \circ \beta^*(q)$ = {q},$\beta^*(r) \circ \beta^*(r)$ = {r},$\beta^*(p) \circ \beta^*(q)$ = {p,q},$\beta^*(p) \circ \beta^*(r)$ = {p,r},$\beta^*(q) \circ \beta^*(r)$ = {q,r}.*

$\circ$	p	q	r
p	$\{p\}$	$\{p,q\}$	$\{p,r\}$
q	$\{p,q\}$	$\{q\}$	$\{q,r\}$
r	$\{p,r\}$	$\{q,r\}$	$\{r\}$

Here $\mathcal{H}$ is a SH and X = {p,q}, then $\tau_{\beta^*}(X) = \{H, \emptyset, \{p,q\}\}$.

Remark 3.4 *:For any set S the smallest equivalence relation is the one that contains all the pairs (s,s) for s ∈ S.*

Theorem 3.5 *:If $\mathcal{H}$ is a SH with β^* is the smallest equivalence relation then $(\mathcal{H}, \tau_{\beta^*}(X))$ is $\mathcal{N}T_1$ space.*

Proof:If a ∈ $\beta^*(a)$ and b ∈ $\beta^*(b)$ and hence $\beta^*(a) \circ \beta^*(b) = \{\beta^*(c) | c \in \beta^*(a) \circ \beta^*(b)\}$ for all a,b ∈ $\mathcal{H}$. Since β^* is smallest equivalence relation it holds both the properties of semihypergroup by Proposition 3.1.$\beta^*(a) \subset$ X and $\beta^*(b) \subset$ X and so X = $L_{\beta^*}(X) = U_{\beta^*}(X)$. Hence we obtained $\mathcal{N}T_1$ space.

Theorem 3.6 *:Let X and Y be nonempty subsets of $\mathcal{H}$. Then the following relatins are satisfied:*

(i) $L_{\beta^*}(X) \subseteq X \subseteq U_{\beta^*}(X)$.

(ii) $U_{\beta^*}(X \cup Y) = U_{\beta^*}(X) \cup U_{\beta^*}(Y)$.

(iii) $X \subseteq Y$ *implies* $L_{\beta^*}(X) \subseteq L_{\beta^*}(Y)$ *and* $U_{\beta^*}(X) \subseteq U_{\beta^*}(Y)$.

Proof*(i):If* $x \in L_{\beta^*}(X)$ *that is* $x \in X$ *implies* $x \in U_{\beta^*}(X)$.
(ii) $x \in U_{\beta^*}(X \cup Y)$. *Then* $x \in x\beta^* \cap (X \cup Y)$.*It follows that* $x \in x\beta^* \cap X$ *and* $x \in x\beta^* \cap Y$ *and hence* $x \in U_{\beta^*}(X)$ *or* $x \in U_{\beta^*}(Y)$ *that is* $x \in U_{\beta^*}(X) \cup U_{\beta^*}(Y)$.
(iii) By (i) $L_{\beta^*}(X) \subseteq X$ *and since* $X \subseteq Y$, $L_{\beta^*}(Y) \subseteq Y$ *hence* $L_{\beta^*}(X) \subseteq L_{\beta^*}(Y)$ *and it is obvious that* $U_{\beta^*}(X) \subseteq U_{\beta^*}(Y)$.

4 Relationship between nano topology and hypergroup

In this section we exhibit the comparison among hypergroup and nano topology as well as hypergroup and topological space.

Definition 4.1 *[12]:Let* $(\mathcal{H}, \tau)$ *be a topological space. Any element x belongs to a least open set O(x) called fundamental open set(FOS).*

Proposition 4.2 *[12]:"If a set $\mathcal{H}$ is equipped with the topology τ and $\circ$ is a hyperoperation on $\mathcal{H}$ defined by x∘y = O(x) ∪ O(y) then the set $\mathcal{H}$ is a hypergroup".*

Proposition 4.3 *[12]: "Let ($\mathcal{H}$,τ) be a topological space. The relation R on $\mathcal{H}$ is defined on H by x R y iff O(y) = O(x) is an equivalence relation".*

Remark 4.4 *: There is an interesting Remark that we can form nano topology from any topological space by using Definition 4.1 and Proposition 4.3 as shown in following example.*

Example 4.5 *: Let $\mathcal{U}$ = {a,b,c}, τ = $\{\mathcal{U}, \emptyset, \{a\}, \{a,b\}\}$ where O(a)={a}, O(b)={a,b} and O(c)= X. Here O(a) = O(a), O(b)= O(b) and O(c) = O(c) and hence the equivalence class {{a},{b},{c}}. Therefore the nano topology when X = {a,b} and $\tau_R(X) = \{\mathcal{U}, \emptyset, \{a,b\}\}$.*

Theorem 4.6 *:If $(\mathcal{H}, \tau)$ is a discrete topological space then $\mathcal{H}$ is a SH.*

Proof:Here $\mathcal{H}$ is discrete then the least open set of each element is singleton set and by Proposition 3.1 implies that the relation is a smallest equivalence relation. Hence it is obvious that $\mathcal{H}$ is a SH by definition.

Definition 4.7 *Let $(\mathcal{H}, \tau_R(X))$ be a nano topological space. Any element x belongs to a nano least open set (NO(x)) is called nano fundamental open set.*

Example 4.8 *:Let $\mathcal{H}$ = {a,b,c}, $\mathcal{H}/R$ = {{a,b},{c}} and X={a,b} then $\tau_R(X) = \{\mathcal{H}, \emptyset, \{a,b\}\}$ and NO(a)={a,b},NO(b)= {a,b}, NO(c) = $\mathcal{H}$ then*

$\circ$	a	b	c
a	$\{a,b\}$	$\{a,b\}$	$\mathcal{H}$
b	$\{a,b\}$	$\{a,b\}$	$\mathcal{H}$
c	$\mathcal{H}$	$\mathcal{H}$	$\mathcal{H}$

Proposition 4.9 *:If a set $\mathcal{H}$ is equipped with the nano topology $\tau_R(X)$ and $\circ$ is a hyperoperation on $\mathcal{H}$ defined by $x \circ y$ = NO(x) $\cup$ NO(y) then the set ($\mathcal{H}$,$\circ$) is a hypergroup. The family {NO(x):x $\in \mathcal{H}$ } is called nano fundamental family.*

Proof:It is clear that x $\circ$ y $\neq \emptyset$. Since union is associative then $\circ$ is also associative. Let a $\in \mathcal{H}$ then for all x $\in \mathcal{H}$,x $\in$ NO(a) $\cup$ NO(x) = a $\circ$ x so $\mathcal{H} \subset \cup_{x \in \mathcal{H}}$(a$\circ$x) = a$\circ\mathcal{H} \subset \mathcal{H}$ and then it is obvious that a$\circ\mathcal{H} = \mathcal{H} = \mathcal{H} \circ$a.

Proposition 4.10 *:If x $\in$ G; where G is an nano open set of the NTS ($\mathcal{H}$, $\tau_R(X)$) then NO(x) $\subset$ G.*

Proof:Since x $\in$ NO(x) $\cap$ G then NO(x) $\cap$ G $\neq \emptyset$. Suppose that NO(x) is not included in G, then since NO(x) $\cap$ G $\nsubseteq$ NO(x) $\cap$ G is also a nonempty nano open set containing x, the nano open set NO(x) is not the least nano open set containing x which is a contradiction.

Theorem 4.11 *:If ($\mathcal{H}$,τ) be a door space then $(\mathcal{H}, \tau_R(X))$ is $\mathcal{N}T_1$ topological space.*

Proof:If $\mathcal{H}$ is a door space then every subset of $\mathcal{H}$ is either open or closed.
case(i) If x belongs to all least open sets then the least open sets are equal to itself,then there exists a smallest equivalence relation and hence H is $\mathcal{N}T_1$ space.
case(ii)If each x belongs to distinct least open sets then there exists smallest equivalence relation and hence proved.

Remark 4.12 *: The nano topology is derived from ISH via fundamental open set is always coarser than nano topology generated by fundamental open set which is shown by following example.*

Example 4.13 *:Let* $\mathcal{H}=\{a,b,c,d\}$,$\tau=\{\mathcal{H},\emptyset,\{a\},\{b,c\},\{a,b,c\},\{d\},\{b,c,d\},\{a,d\}\}$ *and O(a)* $=\{a\}$,*O(b)* $=\{b,c\}$,*O(c)* $=\{b,c\}$,*O(d)* $=\{d\}$ *then*

$\circ$	a	b	c	d
a	$\{a\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,d\}$
b	$\{a,b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c,d\}$
c	$\{a,b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c,d\}$
d	$\{a,d\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\{d\}$

so O(a) = O(a), O(b) = O(b), O(c) = O(c), O(d) = O(d), O(b) = O(c), O(a) $\neq$ *O(b), O(a)* $\neq$ *O(c), O(a)* $\neq$ *O(d),O(b)* $\neq$ *O(d),O(c)* $\neq$ *O(d), therefore* $\mathcal{H}$ */R* $=\{\{a\},\{b,c\},\{d\}\}$ *and A* $=\{a,b,d\}$, $\tau_R(A)=\{\mathcal{H},\emptyset,\{a,d\},\{b,c\}\}$. *Here* $S=\mathcal{H}$, $\mathcal{H}$ */* $R_S=\{a,b,c,d\}$,$\tau_S(A)=\{\mathcal{H},\emptyset\}$. *Here* $L_S(A)\subset L_R(A)$, $U_S(A)\subset U_R(A)$ *and* $B_S(A)\subset B_R(A)$. *Therefore* $\tau_{R_S}(A)\subseteq\tau_R(A)$ *so* $\tau_{R_S}(A)$ *is coarser than* $\tau_R(A)$.

A	$L_R(A)$	$U_R(A)$	$B_R(A)$	$L_S(A)$	$U_S(A)$	$B_S(A)$
$\mathcal{H}$	$\mathcal{H}$	$\mathcal{H}$	$\emptyset$	$\mathcal{H}$	$\mathcal{H}$	$\emptyset$
$\{a\}$	$\{a\}$	$\{a\}$	$\emptyset$	$\emptyset$	$\mathcal{H}$	$\mathcal{H}$
$\{b\}$	$\emptyset$	$\{b,c\}$	$\{b,c\}$	$\emptyset$	$\mathcal{H}$	$\mathcal{H}$
$\{c\}$	$\emptyset$	$\{b,c\}$	$\{b,c\}$	$\emptyset$	$\mathcal{H}$	$\mathcal{H}$
$\{d\}$	$\{d\}$	$\{d\}$	$\emptyset$	$\emptyset$	$\mathcal{H}$	$\mathcal{H}$
$\{a,b\}$	$\{a\}$	$\{a,b,c\}$	$\{b,c\}$	$\emptyset$	$\mathcal{H}$	$\mathcal{H}$
$\{a,c\}$	$\{a\}$	$\{a,b,c\}$	$\{b,c\}$	$\emptyset$	$\mathcal{H}$	$\mathcal{H}$
$\{a,d\}$	$\{a,d\}$	$\{a,d\}$	$\emptyset$	$\emptyset$	$\mathcal{H}$	$\mathcal{H}$
$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	$\emptyset$	$\emptyset$	$\mathcal{H}$	$\mathcal{H}$
$\{b,d\}$	$\{d\}$	$\{b,c,d\}$	$\{b,c\}$	$\emptyset$	$\mathcal{H}$	$\mathcal{H}$
$\{c,d\}$	$\{d\}$	$\{b,c,d\}$	$\{b,c\}$	$\emptyset$	$\mathcal{H}$	$\mathcal{H}$
$\{a,b,c\}$ $\mathcal{H}$	$\{a,b,c\}$	$\{a,b,c\}$	$\emptyset$	$\emptyset$	$\mathcal{H}$	
$\{a,b,d\}$	$\{a,d\}$	$\mathcal{H}$	$\{b,c\}$	$\emptyset$	$\mathcal{H}$	$\mathcal{H}$
$\{a,c,d\}$	$\{a,d\}$	$\mathcal{H}$	$\{b,c\}$	$\emptyset$	$\mathcal{H}$	$\mathcal{H}$
$\{b,c,d\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\emptyset$	$\emptyset$	$\mathcal{H}$	$\mathcal{H}$

5 Characterizations

In this section we derive hypergroup from nano topology and establish its characterizations.

Theorem 5.1 *:A nano open set* $\mathcal{K}$ *of a nano topological space (H,*$\tau_R(X)$*) forms a subhypergroup of* $\mathcal{H}$.

Proof:case(i):If x $\in\mathcal{K}$ then x $\in$ NO(x) $\cap\,\mathcal{K}$ which is nano open set. Consequently, NO(x) $\subset\mathcal{K}$ and then a $\circ$ b $\subset\mathcal{K}$, for all a,b $\in\mathcal{K}$.
case(ii):By the definition of nano topology, for any x $\in\mathcal{K}$ we have x $\in$ NO(x) and for all a $\in\mathcal{K}$, x $\in$ NO(a) $\cup$ NO(x) and therefore $\mathcal{K}\subset\mathcal{K}\circ$ a $\subset\mathcal{K}\Longrightarrow\mathcal{K}=\mathcal{K}\circ$ a. Hence $\mathcal{K}$ is a nano subhypergroup.

Theorem 5.2 *: A nano open set* $\mathcal{K}$ *of a nano topological space (*$\mathcal{H}$ *,*$\tau_R(X)$*) forms a normal subhypergroup of H.*

Proof:By Theorem 5.1, $\mathcal{K}$ is a subhypergroup.If a $\in$ H and b $\in$ H then there exist NO(a) and NO(b). Now a $\circ$ b = NO(a) $\cup$ NO(b) implies that b $\circ$ a = NO(b) $\cup$ NO(a). Therefore a $\circ\,\mathcal{K}$ $=\mathcal{K}\circ$ a.

Definition 5.3 *:Let* $(\mathcal{H}, \tau_R(X))$ *be a nano topological space and NO(a) is nano saturated if for all* $x \in NO(a)$ *we have* $NO(x) = NO(a)$.

Example 5.4 *: Let* $\mathcal{H} = \{a,b,c,d\}$,$X = \{a,c\}$ *and* $\mathcal{H}/R = \{\{a,b,c\},\{d\}\}$ *then* $\tau_R(X) = \{\mathcal{H}, \emptyset, \{a,b,c\}\}$ *where NO(a)=*$\{a,b,c\}$ *and hence NO(a) is nano saturated.*

Proposition 5.5 *:Let* $(\mathcal{H},\tau_R(X))$ *be a nano topology, if the elements of the family* $F_{\tau_R}(X) = \{NO(x)/x \in \mathcal{H}\}$ *are saturated then the family form a partition of* $\mathcal{H}$.

Theorem 5.6 *: Let NO(x) be a nano fundamental family. For any nano open set O and nano saturated open set NO(x) either* $NO(x) \cap O = \emptyset$ *or* $NO(x) \subset O$.

Proof:Suppose NO(x) $\cap$ O $\neq \emptyset$ then for any a $\in$ NO(x) $\cap$ O implies NO(x) = NO(a) and NO(a) $\subset$ O .Therefore NO(x) $\subset$ O.

Definition 5.7 *:For* $a \in \mathcal{H}$ *the mapping* $f_a : \mathcal{H}\backslash\{a\} \longrightarrow P^*(H)$ *where* $f_a(x) = a \circ x$ *is said*

(i) semi injective if $a \circ x = a \circ y \Longrightarrow NO(x) = NO(y)$.

(ii) semi surjective if for all $\emptyset \neq A \subset H$,*there exists* $x \in \mathcal{H}$ *such that* $A \subset f_a(x) = a \circ x$.

Theorem 5.8 *:Let* $(\mathcal{H},\tau_R(X))$ *be a nano topological space and NO(x) be a nano saturated family and nano fundamental family, then the mapping* $f_a : \mathcal{H}\backslash\{a\} \longrightarrow P^*(H)$ *where* $f_a(x) = a \circ x$ *is semi injective.*

Proof:Suppose NO(a) $\cup$ NO(x)= NO(a) $\cup$ NO(y) and b $\in$ NO(x).
(i) If b $\in$ NO(a) then NO(a) = NO(x) and therefore NO(x) = NO(x) $\cup$ NO(y) which implies that NO(y) $\subset$ NO(x) and since y $\in$ NO(y) $\subset$ NO(x) we have NO(y) = NO(x).
(ii) If b $\notin$ NO(a) but b $\in$ NO(a) $\cup$ NO(x) = NO(a) $\cup$ NO(y) then b $\in$ NO(y) and so NO(x) $\subset$ NO(y). Hence NO(x) = NO(y).

Theorem 5.9 *:Let* $(\mathcal{H},\tau_R(X))$ *be a nano topological space and NO(x) be a nano fundamental family, where the union of two fundamental nano open sets is a fundamental nano open set,then the mapping* $f_a : H\backslash\{a\} \longrightarrow P^*(\mathcal{H})$ *where* $f_a(x) = a \circ x$ *is semi surjective.*

Proof:Let A $\in$ $P^*(\mathcal{H})$ then A $\subset \cup_{x\in A}$NO(x) .There exists b $\in \mathcal{H}$ such that NO(b) $\subset \cup_{x\in A} NO(x)$ and A $\subset \cup_{x\in A} NO(x) \implies NO(b) \subset NO(a) \cup NO(b)$.

Conclusion:In this paper, we have introduced nano least open sets and presented their properties. By using the fundamental relations and least open sets we defined the nano topology and characterizations are obtained. Moreover we defined hypergroup from nano topology and its properties are discussed. We believe that these results will contribute to the study of nano topological spaces and will help researchers for further study.

Conflict of Interest: On behalf of all the authors, there is no conflict of interest.
Declaration: I declare that this is my own work. The content of this paper have not published by any authors in journals.

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Chapter 15

A neutrosophic topology for neutrosophic automata

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Abstract

This paper aims to investigate the topological properties of neutrosophic automata through various neutrosophic operators and explore the connections between them. We associate neutrosophic co-topologies and neutrosophic topologies with a given neutrosophic automaton, utilizing these associations to define certain algebraic properties. Lastly, we highlight the intriguing relationships between neutrosophic homomorphisms and the neutrosophic co-topologies/topologies induced by the neutrosophic operators.

Keywords: Neutrosophic automata, neutrosophic succesor operator, neutrosophic core operator, neutrosophic topology, neutrosophic co-topology, neutrosophic homomorphism.

1 INTRODUCTION

In 1965, Zadeh [36] introduced fuzzy set theory, revolutionizing the approach to handling uncertainty and ambiguity in both mathematical models and real-world applications. By allowing elements to have varying degrees of membership rather than being confined to binary categories, fuzzy set theory opened up new possibilities. This innovation paved the way for the "fuzzification" of a wider range of mathematical concepts and other fields. One such development was Atanassov's [1] introduction of intuitionistic fuzzy sets, which not only address the degree of membership but also account for the uncertainty or hesitation present in real-life scenarios. Building on these ideas, Smarandache [28, 29] proposed the neutrosophic set, a framework that incorporates three components—truth, indeterminacy, and falsity—to manage uncertainty, making it particularly valuable

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for situations involving incomplete, inconsistent, or contradictory data.

The algebraic study of automata theory is of fundamental importance [6, 7, 10]. Many researchers have focused on the algebraic analysis of fuzzy automata in various forms (see, e.g., [8, 9, 14, 17, 22, 23, 24, 25, 26, 30, 31, 32]). In [7], concepts like separateness, connectedness, and retrievability of automata were introduced, along with their applications driven by operators. Building upon Bavel's influential contributions [7] to algebraic automata theory, the connection between topology and automata was first explored in [27, 30], where it was shown that several well-established topological concepts could be effectively applied to automata theory, including their properties of connectivity and separation.

Neutrosophic automata, an extension of classical automata theory based on neutrosophic logic, offer a fresh and adaptable framework in computational theory. Neutrosophic logic, developed by Florentin Smarandache [28], extends classical, fuzzy, and intuitionistic logics by addressing uncertainty, imprecision, vagueness, incompleteness, and inconsistency through three degrees: truth (T), indeterminacy (I), and falsity (F). This approach provides a more comprehensive representation of real-world issues, especially those characterized by ambiguity and incomplete data. Wang et al. [33, 34] introduced single-valued neutrosophic sets and interval neutrosophic sets as simpler instances of neutrosophic sets. However, the topological, algebraic, and operator-based exploration of neutrosophic automata remains relatively underdeveloped. The algebraic study is recognized for its importance because the algebraic properties are crucial in advancing the foundations of computer science. Moreover, the use of topological methods in the study of fuzzy sets and neutrosophic automata has proven beneficial in (i) simplifying the arguments used to prove various results and (ii) offering additional insights from the well-established field of topology. The main goal of this work is to introduce and analyze the topological characterization of neutrosophic automata. Neutrosophic operators, such as the neutrosophic source, neutrosophic successor, and neutrosophic core operators, induce neutrosophic topologies and cotopologies based on neutrosophic residuated lattices.

Neutrosophic automata have a wide range of applications, including in control systems [18, 19], algebraic structures [11, 12, 13, 15, 16, 21, 20], and research in topological and metric spaces [2, 3, 4, 5]. The operators discussed in these studies serve to expand and refine earlier algebraic methods in automata theory while also contributing to advancements in topology, financial analysis, and various other interdisciplinary areas.

This paper explores the theory of neutrosophic automata using operator theory, which offers a solid foundation for the algebraic and topological analysis of neutrosophic automata. The structure of the paper is as fol-

lows: Section 2 presents the basic concepts essential for the study. In Section 3, we introduce and thoroughly examine three fundamental operators the neutrosophic source, neutrosophic successor, and neutrosophic core operators for a given neutrosophic automaton, focusing on their algebraic properties. Section 4 presents the concepts of neutrosophic topology and neutrosophic co-topology, investigating their relationships with the operators discussed in Section 3. Finally, Section 5 explores neutrosophic homomorphisms between neutrosophic automata, including an analysis of the behavior of neutrosophic operators under such homomorphisms.

2 BACKGROUND

In this section, we revisit the essential notations and foundational concepts pertaining to neutrosophic sets, encompassing implication operators, neutrosophic residuated lattices, neutrosophic points, and the various properties and operations governing neutrosophic sets. The theoretical underpinnings of neutrosophic sets and their properties are derived from the seminal works of [11, 12, 15, 16, 28], while the distinct properties of neutrosophic automata, implication operators, and fundamental neutrosophic set operations are explored in [20, 21]. The discourse begins by addressing the following key points.

Definition 2.1 *A* **neutrosophic set (NS, in short)** N *on a non-empty set* X *is defined as an entity of the form* $N = \{< u, F_N(u), G_N(u), H_N(u) >: u \in X\}$*, where the functions* $F_N, G_N, H_N : X \to]0^-, 1^+[$ *define respectively the degree of membership (or truth), the degree of indeterminacy and the degree of non-membership (or false) of each element* $u \in X$ *to the set* N *. As, the sum of* $F_N(u), G_N(u), H_N(u)$ *have unrestricted. Consequently, for each* $u \in X, 0^- \leq F_N(u) + G_N(u) + H_N(u) \leq 3^+$*.*

Remark 2.1 *A neutrosophic set* $N = < u, F_N(u), G_N(u), H_N(u) >: u \in X$ *is conventionally represented as an ordered triple* $< F_N, G_N, H_N >$ *in the non-standard unit interval* $]0^-, 1^+[$ *on* X*. The neutrosophic sets (***NSs, in short)** $\mathbf{0_N}$ *and* $\mathbf{1_N}$ *which signify constant NSs in* X *and are defined as* $0_N =< 0, 1, 1 >$ *and* $1_N =< 1, 0, 0 >$*, where* $0, 1 : X \to]0^-, 1^+[$ *are defined respectively by* $\mathbf{0}(\mathbf{u}) = \mathbf{0}$ *and* $\mathbf{1}(\mathbf{u}) = \mathbf{1}$*. The NS* $a = (\alpha, \beta, \gamma)$ *such that* $\widehat{a} = \widehat{(\alpha, \beta, \gamma)}$ *is expressed as* $\widehat{a}(u) = a$ *for all* $u \in X$*, where* α, β*, and* γ *are the* α*-valued,* β*-valued, and* γ*-valued constant neutrosophic sets in* X *respectively, with the condition* $0^- \leq \sigma + \beta + \gamma \leq 3^+$*.*

This paper adopts the interval $[0, 1]$ rather than the notation $]0^-, 1^+[$ for practical purposes, as the latter may present obstacles in real-world applications. Moreover, $\mathbf{NS}(\mathbf{X})$ will signify the collection of all neutrosophic sets within X and I^* denotes the set $\{(u, v, w) : ((u, v, w) \in [0, 1] \times [0, 1] \times [0, 1], 0 \leq u + v + w \leq 3\}$. A neutrosophic set $N =< F_N, G_N, H_N >$ in X will frequently be viewed as a function $N : X \to I^*$, given by $N(u) = \{F_N(u), G_N(u), H_N(u) : u \in X\}$.

To begin, we revisit several foundational properties of neutrosophic sets in X.

Definition 2.2 *For NSs* $N_1 =< F_{N_1}, G_{N_1}, H_{N_1} >, N_2 =< F_{N_2}, G_{N_2}, H_{N_2} >$ *and* $N_i =< F_{N_i}, G_{N_i}, H_{N_i} >, i \in J$ *in* $u \in X$. *We have*

(1) $N_1 \leq N_2$ *if* $F_{N_1}(u) \leq F_{N_2}(u), G_{N_1}(u) \geq G_{N_2}(u)$ *and* $H_{N_1}(u) \geq H_{N_2}(u)$;

(2) $\vee_{i\in J} N_i(u) = (\vee_{i\in J} F_{N_i}(u), \wedge_{i\in J} G_{N_i}(u), \wedge_{i\in J} H_{N_i}(u))$;

(3) $\wedge_{i\in J} N_i(u) = (\wedge_{i\in J} F_{N_i}(u), \vee_{i\in J} G_{N_i}(u), \vee_{i\in J} H_{N_i}(u))$;

(4) $N^c = (1 - F_N, 1 - G_N, 1 - H_N)$;

(5) $0_N \subseteq N \subseteq 1_N; 0_N^c = 1_N$ *and* $1_N^c = 0_N$;

Definition 2.3 *[20] A* **neutrosophic residuated lattice** *is an algebra* $I_N = \{I^*, \wedge, \vee, \otimes, \rightarrow, 0_N, 1_N\}$, *where*

(1) $(I^*, \wedge, \vee, 0_N, 1_N)$ *is a lattice with the least element* 0_N *and greatest element* 1_N,

(2) $(I^*, \otimes, 1_N)$ *is a commutative monoid with unit* 1_N, *and*

(3) $\otimes$ *and* $\rightarrow$ *form an adjoint pair, i.e., for all* $\sigma_N = (\sigma_1, \sigma_2, \sigma_3), \beta_N = (\beta_1, \beta_2, \beta_3), \gamma_N = (\gamma_1, \gamma_2, \gamma_3) \in I^*, \sigma_N \otimes \beta_N \leq \gamma_N \Leftrightarrow \sigma_N \leq \beta_N \rightarrow \gamma_N$, *i.e.,* $\sigma_1 \otimes \beta_1 \leq \gamma_1 \Leftrightarrow \sigma_1 \leq \beta_1 \rightarrow \gamma_1, \sigma_2 \otimes \beta_2 \geq \gamma_2 \Leftrightarrow \sigma_2 \geq \beta_2 \leftarrow \gamma_2$ *and* $\sigma_3 \otimes \beta_3 \geq \gamma_3 \Leftrightarrow \sigma_3 \geq \beta_3 \leftarrow \gamma_3$.

Proposition 2.1 *[20] The* **neutrosophic point (NP, in short)** $x_{(\alpha,\beta,\gamma)}$ *on* X *is a NS in* X *such that for all* $x, y \in X$ *and for all* $(\alpha, \beta, \gamma) \in I^*$,

$$x_{(\alpha,\beta,\gamma)}(y) = \begin{cases} (\alpha, \beta, \gamma) & \text{if } x = y \\ 0_N & \text{if } x \neq y. \end{cases}$$

i.e.,

$$F_{x_\alpha}(y) = \begin{cases} F_\alpha & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

$$G_{x_\beta}(y) = \begin{cases} G_\beta & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

and

$$H_{x_\gamma}(y) = \begin{cases} H_\gamma & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Proposition 2.2 *Let* $I_N = \{I^*, \wedge, \vee, \otimes, \rightarrow, 0_N, 1_N\}$ *be a neutrosophic residuated lattice and* $N = (F_N, G_N, H_N) \in NS(X)$. *Then for all* $\sigma_N = (\sigma_1, \sigma_2, \sigma_3), \beta_N = (\beta_1, \beta_2, \beta_3)$ *and,* $\gamma_N = (\gamma_1, \gamma_2, \gamma_3) \in I^*$ *following properties hold:*

(i) $1_N \rightarrow N = (1, 0, 0) \rightarrow (F_N, G_N, H_N) = (F_N, G_N, H_N) = N$;

(ii) $\sigma_N \leq \beta_N \Longleftrightarrow \sigma_N \rightarrow \beta_N = 1_N$;

(iii) $\beta_N \leq \sigma_N \rightarrow (\sigma_N \otimes \beta_N)$;

(iv) $(\sigma_N \otimes \beta_N) \rightarrow \gamma_N = \sigma_N \rightarrow (\beta_N \rightarrow \gamma_N)$;

(v) $\sigma_N \otimes \wedge_{i\in J} \beta_{N_i} \leq \wedge_{i\in J} (\sigma_N \otimes \beta_{N_i})$;

(vi) $\sigma_N \otimes (\vee_{i\in J} \beta_{N_i}) = \vee_{i\in J} (\sigma_N \otimes \beta_{N_i})$;

(vii) $(\sigma_N \rightarrow \beta_N) \otimes \sigma_N \leq \beta_N$;

(viii) $(\sigma_N \otimes \beta_N) \rightarrow \gamma_N = (\beta_N \otimes \sigma_N) \rightarrow \gamma_N$; *and*

(ix) $\vee_{i\in J}\sigma_{N_i} \to \beta_N = \wedge_{i\in J}(\sigma_{N_i} \to \beta)$.

Definition 2.4 *The negator on I^* is a map $c : I^* \to I^*$ such that $\forall, x = (u, v, w) \in X, x^c = x \to 0_N = (u \to 0, v \leftarrow 1, w \leftarrow 1)$.*

Let $\psi : X \to Y$ be a map.Then according to Zadeh's extension principal in neutrosophic sets, ψ cab be extended to the operator $\psi^{\to} : NS(X) \to NS(Y)$ and $\psi^{\leftarrow} : NS(Y) \to NS(X)$ such that for all $N \in NS(X), N' \in NS(Y)$ and $y \in Y, F_{\overrightarrow{\psi}(N)}(y) = \vee\{F_N(x) : x \in X, F_{\psi(x)} = F_y\}, G_{\overrightarrow{\psi}(N)}(y) = \wedge\{G_N(x) : x \in X, G_{\psi(x)} = G_y\}$ and $H_{\overrightarrow{\psi}(N)}(y) = \wedge\{H_N(x) : x \in X, H_{\psi(x)} = H_y\}$, and
$F_{\overleftarrow{\psi}(N')} = F_{N'o\psi}, G_{\overleftarrow{\psi}(N')} = G_{N'o\psi}$ and, $H_{\overleftarrow{\psi}(N')} = H_{N'o\psi}$.

3 Neutrosophic operators and Neutrosophic automata

The proofs and procedures applied in this section are similar to those discussed in our previous work [20]. In this section, we introduce and study the concepts of neutrosophic operators, namely neutrosophic source, neutrosophic successor and neutrosophic core operators for a given neutrosophic automata based on neutrosophic residuated lattices. Next, we discuss the relationships between thses operators and some algebraic properties of neutrosophic automata.

Definition 3.1 *[21] A **Neutrosophic Automaton, (NAut, in short)** is a triple $A = (S, X, \delta^n)$, where S and X are non-empty sets, called the set of states and the set of inputs respectively. The neutrosophic transition function $\delta^n = (F_\delta, G_\delta, H_\delta)$ be a neutrosophic subset of $S \times X \times S$, i.e a map $\delta^n : S \times X \times S \to I^*$.*

Remark 3.1 *Let X^* denote the free monoid generated by the set X, with e being its identity. The extension of δ is denoted as $\delta^{*n} = (F_{\delta^*}, G_{\delta^*}, H_{\delta^*}) : S \times X^* \times S \to I^*$. This extension is characterized by the property that for any $s_1, s_2 \in S, u \in X^*$, and $x \in X$, the following holds:*

$$F_{\delta^*}(s_1, e, s_2) = \begin{cases} 1 & \text{if } s_1 = s_2 \\ 0 & \text{if } s_1 \neq s_2, \end{cases}$$

$$G_{\delta^*}(s_1, e, s_2) = H_{\delta^*}(s_1, e, s_2) = \begin{cases} 0 & \text{if } s_1 = s_2 \\ 1 & \text{if } s_1 \neq s_2 \end{cases}$$

$F_{\delta^*}(s_1, ux, s_2) = \vee\{F_{\delta^*}(s_1, u, s_3) \otimes F_\delta(s_3, x, s_2) : s_3 \in Q\}, G_{\delta^*}(s_1, ux, s_2) = \wedge\{G_{\delta^*}(s_1, u, s_3) \otimes G_\delta(s_3, x, s_2) : s_3 \in Q\}$, and $H_{\delta^*}(s_1, ux, s_2) = \wedge\{H_{\delta^*}(s_1, u, s_3) \otimes H_\delta(s_3, x, s_2) : s_3 \in Q\}$.

Lemma 3.1 *Let $A = (S, X, \delta^N)$ be a NAut. Then $\forall s_1, s_2 \in S$ and $\forall u, v \in X^*$, $F_{\delta^*}(s_1, uv, s_2) = \vee\{F_{\delta^*}(s_1, u, s_3) \otimes F_{\delta^*}(s_3, v, s_2) : s_3 \in S\}$, $G_{\delta^*}(s_1, uv, s_2) = \wedge\{G_{\delta^*}(s_1, u, s_3) \otimes G_{\delta^*}(s_3, v, s_2) : s_3 \in S\}$ and $H_{\delta^*}(s_1, uv, s_2) = \wedge\{H_{\delta^*}(s_1, u, s_3) \otimes H_{\delta^*}(s_3, v, s_2) : s_3 \in S\}$*

Proof:- The proof is similar as given in [20].

The existing studies of neutrosophic automata [21] is differs from this neutrosophic automata, as the set of states may be infinite and the input set may be a monoid rather than just a set.

Definition 3.2 *[20] A NAut is a triple $A = (S, X, \delta^n)$, where S is a non-empty set (of states of A), X is a monoid (the input monoid of A), whose identity shall be denoted as e. The neutrosophic transition function $\delta^n = (F_\delta, G_\delta, H_\delta)$ be a neutrosophic subset of $S \times X \times S$, i.e a map $\delta^n : S \times X \times S \to I^*$, such that for all $s_1, s_2 \in S$ and $x, y \in X$,*

$$F_\delta(s_1, e, s_2) = \begin{cases} 1 & \text{if } s_1 = s_2 \\ 0 & \text{if } s_1 \neq s_2, \end{cases}$$

$$G_\delta(s_1, e, s_2) = H_\delta(s_1, e, s_2) = \begin{cases} 0 & \text{if } s_1 = s_2 \\ 1 & \text{if } s_1 \neq s_2 \end{cases}$$

$F_\delta(s_2, xy, s_1) = \vee\{F_\delta(s_2, x, s_3) \otimes F_\delta(s_3, y, s_2) : s_3 \in Q\}, G_\delta((s_2, xy, s_1)) = \wedge\{G_\delta(s_1, x, s_3) \otimes G_\delta(s_3, y, s_2) : s_3 \in Q\}$, and $H_\delta(s_2, xy, s_1) = \wedge\{H_\delta(s_2, x, s_3) \otimes H_\delta(s_3, y, s_2) : s_3 \in Q\}$.

Definition 3.3 *Let $A = (S, X, \delta^n)$ be a NAut and $N = (F_N, G_N, H_N) \in NS(S)$. The* **Neutrosophic source (NSo, in short)** *and the* **Neutrosophic successor (NSu, in short)** *of N are defined as:*
$So(F_N)(s_2) = \vee\{F_N(s_1) \otimes F_\delta(s_2, u, s_1) : s_1 \in S, u \in X\}$,
$So(G_N)(s_2) = \wedge\{G_N(s_1) \otimes G_\delta(s_2, u, s_1) : s_1 \in S, u \in X\}$,
$So(H_N)(s_2) = \wedge\{H_N(s_1) \otimes H_\delta(s_2, u, s_1) : s_1 \in S, u \in X\}$, *and*
$Su(F_N)(s_2) = \vee\{F_N(s_1) \otimes F_\delta(s_1, u, s_2) : s_1 \in S, u \in X\}$
$Su(G_N)(s_2) = \wedge\{G_N(s_1) \otimes G_\delta(s_1, u, s_2) : s_1 \in S, u \in X\}$
$Su(H_N)(s_2) = \wedge\{H_N(s_1) \otimes H_\delta(s_1, u, s_2) : s_1 \in S, u \in X\}, \quad \forall s_2 \in S.$

Remark 3.2 *In this paper, we can use the NSo and NSu opeartor as $So^n(N)(s_2) = \vee\{N(s_1) \otimes \delta^n(s_2, u, s_1) : s_1 \in S, u \in X\}$ and $Su^n(N)(s_2) = \vee\{N(s_1) \otimes \delta^n(s_1, u, s_2) : s_1 \in S, u \in X\}$.*

Proposition 3.1 *Let $A = (S, X, \delta^n)$ be a NAut and $So^n, Su^n : NS(S) \to NS(S)$ be the induced NSo and NSu operators, respectively. Then $\forall N_1, N_2$, $\{(N_j) : j \in J\} \in NS(S)$,*

(i) $Su^n(\widehat{\alpha, \beta, \gamma}) = (\widehat{\alpha, \beta, \gamma})$ *and* $So^n(\widehat{\alpha, \beta, \gamma}) = (\widehat{\alpha, \beta, \gamma}), \forall (\alpha, \beta, \gamma) \in I^*$;

(ii) if $N_1 \leq N_2$ *then* $Su^n(N_1) \leq Su^n(N_2)$ *and* $So^n(N_1) \leq So^n(N_2)$;

(iii) $N_1 \leq Su^n(N_1)$ *and* $N_1 \leq So^n(N_1)$;

(iv) $Su^n(\cup\{N_j : j \in J\}) = \cup\{Su^n(N_j : j \in J)\}$ *and* $So^n(\cup\{N_j : j \in J\}) = \cup\{(So^n(N_j : j \in J)\}$;

(v) $Su^n(Su^n(N_1)) = Su^n(N_1)$ *and* $So^n(So^n(N_1)) = So^n(N_1)$;

(vi) $Su^n(\cap\{N_j : j \in J\}) \leq \cap\{Su^n(N_j : j \in J)\}$ *and* $So^n(\cap\{N_j : j \in J\}) \leq \cap\{So^n(N_j : j \in J)\}$.

Proof: The proof is similar as given in [20].

Definition 3.4 *Let $A = (S, X, \delta^n)$ be a NAut. Then $N = (F_N, G_N, H_N) \in NS(S)$ is called a* **neutrosophic subsystem** *of A if $Su^n(N) \leq N$.*

Definition 3.5 *Let $A = (S, X, \delta^n)$ be a NAut and $N \in NS(S)$. The* **neutrosophic core (NCOR, in short)** *of N is given as*
$\mu^n(N)(s_2) = (F_{\mu(N)}, G_{\mu(N)}, H_{\mu(N)})(s_2)$, *where*
$F_{\mu(N)}(s_2) = \wedge\{F_\delta(s_1, u, s_2) \to F_N(s_1) : s_1 \in S, u \in X\}$;
$G_{\mu(N)}(s_2) = \vee\{G_\delta(s_1, u, s_2) \leftarrow G_N(s_1) : s_1 \in S, u \in X\}$, and
$H_{\mu(N)}(s_2) = \vee\{H_\delta(s_1, u, s_2) \leftarrow H_N(s_1) : s_1 \in S, u \in X\}, \forall s_2 \in S$.

Remark 3.3 *The NCOR can also be written as $\mu^n(N)(s_2) = \wedge\{\delta^n(s_1, u, s_2) \to N(s_1) : s_1 \in S, u \in X\}$.*

Proposition 3.2 *Let $A = (S, X, \delta^n)$ be a NAut and $\mu^n : NS(S) \to NS(S)$ be the induced NCOR operator. Then for all $N_1, N_2, \{N_j : j \in J\} \in NS(S)$,*

(i) $\mu^n(\widehat{\alpha, \beta, \gamma}) = (\widehat{\alpha, \beta, \gamma}), \forall(\alpha, \beta, \gamma) \in I^*$;

(ii) if $N_1 \leq N_2$ *then* $\mu^n(N_1) \leq \mu^n(N_2)$;

(iii) $\mu^n(N_1) \leq N_1$;

(iv) $\mu^n(\cap\{N_j : j \in J\}) = \cap\{\mu^n(N_j : j \in J)\}$;

(v) $\mu^n(\mu^n(N_1)) = \mu^n(N_1)$;

(vi) $\cup\{\mu^n(N_j : j \in J)\} \leq \mu^n(\cup\{N_j : j \in J\}$.

Proof: The proof is similar as given in [20].

4 Neutrosophic topology and neutrosophic co-topology

In this section, firstly we introduce the concepts of strong Alexandrov neutrosophic co-topology/neutrosophic topology and demonstrate that the neutrosophic operators are Alexandroff Kuratowski neutrosophic interior/closure operators. Further, we show that the neutrosophic topology induced by neutrosophic core operator and the neutrosophic co-topologies induced by neutrosophic source and successor operators on the state set of given neutrosophic automata. Next, we illustrate that the induced neutrosophic co-topologies/topologies can be used to characterize the algebraic properties associated with neutrosophic automata.

Definition 4.1 *A map $C^n : N(X) \to N(X)$ is called a* **Kuratowski neutrosophic closure operator, (KNCOP, in short)** *on X if it satisfying $\forall(\alpha, \beta, \gamma) \in I^*, N_1, N_2, N_j \in NS(X), j \in J$,*

(1) $C^n((\widehat{\alpha, \beta, \gamma})) = (\widehat{\alpha, \beta, \gamma})$;

(2) $N_1 \leq C^n(N_1)$;

(3) $C^n(N_1 \cup N_2) = C^n(N_1) \cup C^n(N_2)$; *and*

(4) $C^n(C^n(N_1)) = C^n(N_1)$.

Further, KNCOP $C^n : N(X) \to N(X)$ is called

(5) Alexandroff if $C^n(\vee\{N_j : j \in J\}) = \vee\{C^n(N_j) : j \in J\}$, *and*

(6) strong if for all $(\alpha,\beta,\gamma) \in I^*, C^n(\widehat{(\alpha,\beta,\gamma)} \otimes N_1) = (\alpha,\beta,\gamma) \otimes C^n(N_1)$.

Remark 4.1 *Every KNCOP* C^n *on a set* X *gives rise to a neutrosophic co-topology (NCOT, in short) on* X*, where the NS* N *is neutrosophic closed if and only if* $C^n(N) = N$.

Definition 4.2 *A map* $I^n : N(X) \to N(X)$ *is called a* **Kuratowski neutrosophic interior operator, (KNIOP, in short)** *on* X *if it satisfying* $\forall(\alpha,\beta,\gamma) \in I^*, N_1, N_2, N_j \in NS(X), j \in J$,

(1) $I^n(\widehat{(\alpha,\beta,\gamma)}) = \widehat{(\alpha,\beta,\gamma)}$;

(2) $I^n(N_1) \leq N_1$;

(3) $I^n(N_1 \cap N_2) = I^n(N_1) \cap I^n(N_2)$; *and*

(4) $I^n(I^n(N_1)) = I^n(N_1)$.

Further, KNIOP $I^n : N(X) \to N(X)$ *is called*

(5) Alexandroff if $I^n(\wedge\{N_j : j \in J\}) = \wedge\{I^n(N_j) : j \in J\}$, *and*

(6) strong if for all $(\alpha,\beta,\gamma) \in I^*, I^n(\widehat{(\alpha,\beta,\gamma)} \to N_1) = (\alpha,\beta,\gamma) \to I^n(N_1)$.

Remark 4.2 *Every KNIOP* I^n *on a set* X *gives rise to a neutrosophic topology (NTO, in short) on* X*, where the NS* N *is neutrosophic open if and only if* $I^n(N) = N$.

Definition 4.3 *A NCOT* η^n*, on a nonempty set* X *is a family of NSs in* X *which contains all constant NSs and is neutrosophic closed under arbitrary intersections and finite unions. The NSs in* η^n *are called neutrosophic closed, and the couple* (X,η^n) *is called a neutrosophic co-topological space (NCOTS, in short). Further, the NCOT* η^n *is called (i) Alexandroff if an arbitrary union of neutrosophic closed sets is neutrosophic closed, and (ii) strong if* $\widehat{(\alpha,\beta,\gamma)} \otimes N \in \eta^n, \forall(\alpha,\beta,\gamma) \in I^*$ *and* $N \in \eta^n$.

Definition 4.4 *A NTO* τ^n*, on a nonempty set* X *is a family of NSs in* X *which contains all constant NSs and is neutrosophic closed under arbitrary unions and finite intersections. The NSs in* τ^n *are called neutrosophic open, and the couple* (X,τ^n) *is called a neutrosophic topological space (NTOS, in short). Further, the NTO* τ^n *is called (i) Alexandroff if an arbitrary intersection of neutrosophic open sets is neutrosophic open, and (ii) strong if* $\widehat{(\alpha,\beta,\gamma)} \to N \in \tau^n, \forall(\alpha,\beta,\gamma) \in I^*$ *and* $N \in \tau^n$.

Proposition 4.1 *Let* $A = (S, X, \delta^n)$ *be a NAut. Then,*

(1) the NSu and the NSo operators $Su^n : NS(S) \to NS(S)$ *and* $So^n : NS(S) \to NS(S)$ *is Alexendroff Kuratowski neutrosophic closure operators (AKNCO, in short) on* S.

(2) These two operators induce two NCOT on S*, which we shall denote by* $\eta^n(S)$ *and* $\eta'^n(S)$ *respectively, where* $\eta^n(S) = \{N \in NS(S) : Su^n(N) = N\}$, *and* $\eta'^n(S) = \{N \in NS(S) : So^n(N) = N\}$.

(3) The NCOT $\eta^n(S)$ *and* $\eta'^n(S)$ *are dual in the sense that* $N \in NS(S)$ *is* $\eta^n(S) - NCO$ *iff* N^c *is* $\eta'^n(S) - NCO$.

Proof: Follows from the Definition 3.3 and Proposition 3.1.

Proposition 4.2 *The NCOT $\eta^n(S)$ and $\eta'^n S$ are (i) Alexendroff and (ii) Strong.*

Proof: We have to prove only for $\eta^n(S)$. The proof for the η'^n can be given analogously.

(i) Let $N_j \in \eta^n(S), \forall j \in J$. Then, $Su^n(N_j) = N_j$, i.e. $F_{Su(N_j)} = F_{N_j}; G_{Su(N_j)} = G_{N_j}$ and $H_{Su(N_j)} = H_{N_j}$. Now,
$F_{Su(N_j)} = F_{N_j} \Rightarrow F_{Su(\vee\{N_j:j\in J\})} = \vee F_{Su(\{N_j:j\in J\})} = \vee F_{\{N_j:j\in J\}}; G_{Su(N_j)} = G_{N_j} \Rightarrow G_{Su(\wedge\{N_j:j\in J\})} = \wedge G_{Su(\{N_j:j\in J\})} = \wedge G_{\{N_j:j\in J\}}$, and $H_{Su(N_j)} = H_{N_j} \Rightarrow H_{Su(\wedge\{N_j:j\in J\})} = \wedge H_{Su(\{N_j:j\in J\})} = \wedge H_{\{N_j:j\in J\}}$. Hence $\vee\{N_j : j \in J\} \in \eta^n(S)$. Thus, $\eta^n(S)$ is an Alexendroff NCOT.

(ii) Let $N \in \eta^n(S)$ and $(\alpha, \beta, \gamma) \in I^*$. Then, $Su^n(N) = N$. Now,
$Su^n(N) = N \Rightarrow F_{Su((\alpha)\otimes N)}(s_2) = \vee\{((\alpha) \otimes F_N)(s_1) \otimes F_\delta(s_1, u, s_2) : s_1 \in S, u \in X\} = \alpha \otimes \vee\{F_N(s_1) \otimes F_\delta(s_1, u, s_2) : s_1 \in S, u \in X\} = \alpha \otimes F_{Su(N)}(s_2) = \alpha \otimes F_N(s_2); G_{Su((\beta)\otimes N)}(s_2) = \wedge\{((\beta) \otimes G_N)(s_1) \otimes G_\delta(s_1, u, s_2) : s_1 \in S, u \in X\} = \beta \otimes \wedge\{G_N(s_1) \otimes G_\delta(s_1, u, s_2) : s_1 \in S, u \in X\} = \beta \otimes G_{Su(N)}(s_2) = \beta \otimes G_N(s_2)$. Similarly, $H_{Su((\gamma)\otimes N)}(s_2) = \gamma \otimes H_N(s_2)$. Hence $(\alpha, \beta, \gamma) \otimes N \in \eta^n(S)$. Thus $\eta^n(S)$ is strong NCOT.

Proposition 4.3 *Let (S, X, δ^n) be a NAut.Then*

1. *The neutrosophic core viewed as function $\mu^n : NS(S) \to NS(S)$, turns out to be the Alexandrov Kuratowski neutrosophic interior operator (AKNIO, in short) on S.*
2. *The operator μ^n induce a NTO on S, i.e. $\tau^n(S) = \{N \in NS(S) : \mu^n(N) = N\}$ on S.*

Proof: Follows from Definition 3.5 and Proposition 3.2.

Proposition 4.4 *The NTO on S given by the AKNIO μ^n is precisely $\eta'^n(S)$.*

Proof: We have to prove that $\mu^n(N) = N$ iff $So^n(N) = N$, i.e. $F_{\mu(N)} = F_N$ iff $F_{So(N)} = F_N; G_{\mu(N)} = G_N$ iff $G_{So(N)} = G_N$ and $H_{\mu(N)} = H_N$ iff $H_{So(N)} = H_N, \forall N \in NS(S)$.
Firstly, let $F_{So(N)} = F_N; G_{So(N)} = G_N$ and $H_{So(N)} = H_N$. Then,
$F_{So(N)} = F_N \Rightarrow F_{So(N)}(s_2) = F_N(s_2) \Rightarrow \vee\{F_N(s_1) \otimes F_\delta(s_2, u, s_1) : s_1 \in S, u \in X\} = F_N(s_2) \Rightarrow F_N(s_1) \otimes F_\delta(s_2, u, s_1) \leq F_N(s_2), \forall s_1 \in S, u \in X$
$\Rightarrow F_N(s_1) \leq F_\delta(s_2, u, s_1) \to F_N(s_2), \forall s_1 \in S, u \in X$
$\Rightarrow F_N(s_1) \leq \wedge\{F_\delta(s_2, u, s_1) \to F_N(s_2) : s_1 \in S, u \in X\}$
$\Rightarrow F_N(s_1) \leq F_{\mu(N)}(s_1)$;

$G_{So(N)} = G_N \Rightarrow G_{So(N)}(s_2) = G_N(s_2) \Rightarrow \wedge\{G_N(s_1) \otimes G_\delta(s_2, u, s_1) : s_1 \in S, u \in X\} = G_N(s_2) \Rightarrow G_N(s_1) \otimes G_\delta(s_2, u, s_1) \geq G_N(s_2), \forall s_1 \in S, u \in X$
$\Rightarrow G_N(s_1) \geq G_\delta(s_2, u, s_1) \leftarrow G_N(s_2), \forall s_1 \in S, u \in X$
$\Rightarrow G_N(s_1) \geq \vee\{G_\delta(s_2, u, s_1) \leftarrow G_N(s_2) : s_1 \in S, u \in X\}$
$\Rightarrow G_N(s_1) \geq G_{\mu(N)}(s_1)$. Similarly, $H_{So(N)} = H_N \Rightarrow H_N(s_1) \geq H_{\mu(N)}(s_1)$. Also, as $F_{\mu(N)}(s_1) \leq F_N(s_1); G_{\mu(N)}(s_1) \geq G_N(s_1)$ and $H_{\mu(N)}(s_1) \geq H_N(s_1)$. Thus, we have $\mu^n(N) = N$.

Conversely, let $\mu^n(N) = N$.Then,
$F_{\mu(N)} = F_N \Rightarrow F_{\mu(N)}(s_2) = F_N(s_2) \Rightarrow \wedge\{F_\delta(s_1, u, s_2) \to F_N(s_1) : s_1 \in S, u \in X\} = F_N(s_2) \Rightarrow F_\delta(s_1, u, s_2) \to F_N(s_1) \geq F_N s_2, \forall s_1 \in S, u \in X$
$\Rightarrow F_N(s_2) \leq F_\delta(s_1, u, s_2) \to F_N(s_1), \forall s_1 \in S, u \in X$
$\Rightarrow F_N(s_2) \otimes F_\delta(s_1, u, s_2) \leq F_N(s_1), \forall s_1 \in S, u \in X$
$\Rightarrow \vee\{F_N(s_2) \otimes F_\delta(s_1, u, s_2) : s_1 \in S, u \in X\} \leq F_N(s_1)$
$\Rightarrow F_{So(N)}(s_1) \leq F_N s_1$;

$G_{\mu(N)} = G_N \Rightarrow G_{\mu(N)}(s_2) = G_N(s_2) \Rightarrow \vee\{G_\delta(s_1, u, s_2) \leftarrow G_N(s_1) : s_1 \in S, u \in X\} = G_N(s_2) \Rightarrow G_\delta(s_1, u, s_2) \leftarrow G_N(s_1) \geq G_{s_2}, \forall s_1 \in S, u \in X$
$\Rightarrow G_N(s_2) \geq G_\delta(s_1, u, s_2) \leftarrow G_N(s_1), \forall s_1 \in S, u \in X$
$\Rightarrow G_N(s_2) \otimes G_\delta(s_1, u, s_2) \geq G_N(s_1), \forall s_1 \in S, u \in X$
$\Rightarrow \wedge\{G_N(s_2) \otimes G_\delta(s_1, u, s_2) : s_1 \in S, u \in X\} \geq G_N(s_1)$
$\Rightarrow G_{So(N)}(s_1) \geq G_N s_1$. Similarly, $H_{So(N)}(s_1) \geq H_N(s_1)$.
Also, as $F_N(s_1) \leq F_{So(N)}(s_1); G_N(s_1) \geq G_{So(N)}(s_1)$ and $H_N(s_1) \geq H_{So(N)}(s_1)$.
Thus, $So^N = N$. Hence, $\mu^n(N) = N$ iff $So^N = N$.

Proposition 4.5 *Let $A = (S, X, \delta^n)$ be a NAut. Then, for $N \in NS(S)$ the following assertions are equivalent:*

(1) N is a neutrosophic subsystem of A.

(2) N is $\eta^n(S)$ closed.

Proof: It is obvious from the definition of neutrosophic subsystem and Proposition 4.1.

5 Neutrosophic homomorphisms

In this section, we introduce and study the concept of a neutrosophic homomorphism between two NAut and discuss the behaviour of neutrosophic subsystem and neutrosophic operators introduced in previous sections. Also, we find the neutrosophic continuous between NCOTS/NTOS. We begin with the following concept of neutrosophic homomorphism and strong neutrosophic homomorphism.

Definition 5.1 *Let $A = (S, X, \delta^n)$ and $A' = (R, Y, \lambda^n)$ be the NAut. A* **neutrosophic homomorphism (NHOM, in short)** *ψ between A and A' is a maps $\psi : S \to R$ and $g : X \to Y$, such that $\forall s_1, s_2 \in S$ and $u \in X, \delta^n(s_2, u, s_1) \leq \lambda^n(\psi(s_2), g(u), \psi(s_1))$, i.e. $F_\delta(s_2, u, s_1) \leq F_\lambda(\psi(s_2), g(u), \psi(s_1)); G_\delta(s_2, u, s_1) \geq G_\lambda(\psi(s_2), g(u), \psi(s_1))$ and $H_\delta(s_2, u, s_1) \geq H_\lambda(\psi(s_2), g(u), \psi(s_1))$.*

Further, (ψ, g) is called a **strong neutrosophic homomorphism (SNHOM, in short)** *if $\forall s_1, s_2 \in S$ and $u \in X, \lambda^n(\psi(s_2), g(u), \psi(s_1)) = \vee\{\delta^n(s_2, u, t) : t \in S, \psi(t) = \psi(s_1)\}$.*

The class of all *NAut* and their NHOM form a category under obvious composition of maps. We shall denote it by NAUT.

Definition 5.2 *Let (X, τ_1^n) and (Y, τ_2^n) be the NCOTS/NTOS. A map $\psi : (X, \tau_1^n) \to (Y, \tau_2^n)$ is called a* **neutrosophic continuous (NCON, in short)** *if for each $N \in \tau_2^n$, $\overleftarrow{\psi}^n(N) \in \tau_1^n$, i.e $(F_{\overleftarrow{\psi}(N)}, G_{\overleftarrow{\psi}(N)}, H_{\overleftarrow{\psi}(N)}) \in \tau_1$.*

Proposition 5.1 *If $\psi : A \to A'$ is a NHOM between two NAut $A = (S, X, \delta^n)$ and $A' = (R, X, \lambda^n)$, then $\psi : (S, \eta^n(S)) \to (R, \eta^n(R))$ and $\psi' : (S, \eta'^n(S)) \to (R, \eta'^n(R))$ are NCON.*

Proof: To prove that $\psi : (S, \eta^n(S)) \to (R, \eta^n(R))$ is a NCON, it is enough to show that $N \in \eta^n(R) \Rightarrow \overleftarrow{\psi}^n(N) \in \eta^n(S)$.
For which, let $N = (F_N, G_N, H_N) \in \eta(R)$. Then, $F_S u(N) = F_N; G_S u(N) = G_N$ and $H_S u(N) = H_N$. Now, for all $s_2 \in S$,
$F_{Su(\overleftarrow{\psi}(N))}(s_2) = \vee\{F_{\overleftarrow{\psi}(N)}(s_1) \otimes F_\delta(s_1, u, s_2) : s_1 \in S, u \in X\} = \vee\{F_N(\psi(s_1)) \otimes F_\delta(s_1, u, s_2) : s_1 \in S, u \in X\} \leq \vee\{F_N(\psi(s_1)) \otimes F_\lambda(\psi(s_1), u, \psi(s_2)) : s_1 \in S, u \in X\} = F_S u(N)(\psi(s_2)) = F_N(\psi(s_2)) = F_{\overleftarrow{\psi}(N)}(s_2); G_{Su(\overleftarrow{\psi}(N))}(s_2) = \wedge\{G_{\overleftarrow{\psi}(N)}(s_1) \otimes G_\delta(s_1, u, s_2) : s_1 \in S, u \in X\} = \wedge\{G_N(\psi(s_1)) \otimes G_\delta(s_1, u, s_2) : s_1 \in S, u \in X\} \geq \wedge\{G_N(\psi(s_1)) \otimes G_\lambda(\psi(s_1), u, \psi(s_2)) : s_1 \in S, u \in X\} = G_S u(N)(\psi(s_2)) = G_N(\psi(s_2)) = G_{\overleftarrow{\psi}(N)}(s_2)$. Similarly, $H_{Su(\overleftarrow{\psi}(N))}(s_2) \geq H_{\overleftarrow{\psi}(N)}(s_2)$. Hence, $Su^n(\overleftarrow{\psi}^n(N)) \leq \overleftarrow{\psi}^n(N)$, which implies that $\overleftarrow{\psi}^n(N) \in \eta^n(S)$. Thus, $\psi : (S, \eta^n(S)) \to (R, \eta^n(R))$ is a NCON.

The proof for $\psi' : (S, \eta'^n(S)) \to (R, \eta'^n(R))$ can be given analogously.

The following is towards the behaviour of neutrosophic subsystem under SNHOM.

Proposition 5.2 *Let $\psi : A \to A'$ be an onto SNHOM between two NAut $A = (S, X, \delta^n)$ and $A' = (R, X, \lambda^n)$ and $N \in NS(S)$ be a neutrosophic subsystem. Then (i) $Su^n(\overrightarrow{\psi}^n(N)) = \overrightarrow{\psi}^n(N)$, and (ii) $So^n(\overrightarrow{\psi}^n(N)) = \overrightarrow{\psi}^n(N)$.*

Proof: We prove only (i). Let $N \in NS(S)$ and $s_3 \in R$. Then,
$F_{Su(\overrightarrow{\psi}(N))}(s_3) = \vee\{F_{\overrightarrow{\psi}(N)}(s_4) \otimes F_\lambda(s_4, u, s_3) : s_4 \in R, u \in X\}$
$= \vee\{\vee\{F_N(s_2) \otimes F_\lambda(s_4, u, s_3) : \psi(s_2) = s_4, s_2 \in S\} : s_4 \in R, u \in X\}$
$= \vee\{F_N(s_2) \otimes F_\lambda(\psi(s_2), u, \psi(s_1)) : s_2 \in S, u \in X\}$
$= \vee\{F_N(s_2) \otimes F_\delta(s_2, u, s_1) : s_2 \in S, u \in X\}$
$= \vee\{F_{Su(N)}(s_2) : s_2 \in S, \psi(s_2) = s_3\}$
$= \vee\{F_N(s_2) : s_2 \in S, \psi(s_2) = s_3\} = F_{\overrightarrow{\psi}(N)}(s_3)$;
$G_{Su(\overrightarrow{\psi}(N))}(s_3) = \wedge\{G_{\overrightarrow{\psi}(N)}(s_4) \otimes G_\lambda(s_4, u, s_3) : s_4 \in R, u \in X\}$
$= \wedge\{\wedge\{G_N(s_2) \otimes G_\lambda(s_4, u, s_3) : \psi(s_2) = s_4, s_2 \in S\} : s_4 \in R, u \in X\}$
$= \wedge\{G_N(s_2) \otimes G_\lambda(\psi(s_2), u, \psi(s_1)) : s_2 \in S, u \in X\}$
$= \wedge\{G_N(s_2) \otimes G_\delta(s_2, u, s_1) : s_2 \in S, u \in X\}$
$= \wedge\{G_{Su(N)}(s_2) : s_2 \in S, \psi(s_2) = s_3\}$
$= \wedge\{G_N(s_2) : s_2 \in S, \psi(s_2) = s_3\} = G_{\overrightarrow{\psi}(N)}(s_3)$. Similarly, $H_{Su(\overrightarrow{\psi}(N))}(s_3) = H_{\overrightarrow{\psi}(N)}(s_3)$. Hence, $Su^n(\overrightarrow{\psi}^n)(N) = \overrightarrow{\psi}^n(N)$.

The last section is toward the behaviour of neutrosophic operators under a NHOM.

Proposition 5.3 *Let $\psi : A \to A'$ be a NHOM between two NAut $A = (S, X, \delta^n)$ and $A' = (R, X, \lambda_n)$ with $N \in NS(S)$. Then, (i) $Su^n(\overleftarrow{\psi}^n(N)) \leq \overleftarrow{\psi}^n(Su^n(N))$, and (ii) $So^n(\overleftarrow{\psi}^n(N)) \leq \overleftarrow{\psi}^n(So^n(N))$.*

Proof: We prove only (i). For all $s_2 \in S$.
$F_{Su(\overleftarrow{\psi}(N))}(s_2) = \vee\{F_{\overleftarrow{\psi}(N)}(s_1) \otimes F_\delta(s_1, u, s_2) : s_1 \in S, u \in X\}$

$= \vee\{F_N(\psi(s_1)) \otimes F_\delta(s_1, u, s_2) : s_1 \in S, u \in X\}$
$\leq \vee\{F_N(\psi(s_1)) \otimes F_\lambda(\psi(s_1), u, \psi(s_2) : s_1 \in S, u \in X\}$
$= F_{Su(N)}(\psi(s_2)) = F_{\overleftarrow{\psi}(Su(N))}(s_2)$;
$G_{Su(\overleftarrow{\psi}(N)}(s_2) = \wedge\{G_{\overleftarrow{\psi}(N)}(s_1) \otimes G_\delta(s_1, u, s_2) : s_1 \in S, u \in X\}$
$= \wedge\{G_N(\psi(s_1)) \otimes G_\delta(s_1, u, s_2) : s_1 \in S, u \in X\}$
$\geq \wedge\{G_N(\psi(s_1)) \otimes G_\lambda(\psi(s_1), u, \psi(s_2)) : s_1 \in S, u \in X\}$
$= G_{Su(N)}(\psi(s_2)) = G_{\overleftarrow{\psi}(Su(N))}(s_2)$. Similarly, $H_{Su(\overleftarrow{\psi}(N)}(s_2) \geq H_{\overleftarrow{\psi}(Su(N))}(s_2)$.
Hence, $Su^n(\overleftarrow{\psi^n}(N))(s_2) \leq \overleftarrow{\psi^n}(Su^n(N))(s_2)$.

Proposition 5.4 *Let $\psi : A \to A'$ be a NHOM between two NAut $A = (S, X, \delta^n)$ and $A' = (R, X, \lambda^n)$ with $N \in NS(R)$. Then, $\mu^n(\overleftarrow{\psi^n}(N)) \leq \overleftarrow{\psi^n}(\mu^n(N))$.*

Proof: Similar as Proposition 5.3.

Proposition 5.5 *Let $\psi : A \to A'$ be a NHOM between two NAut $A = (S, X, \delta^n)$ and $A' = (R, X, \lambda^n)$ with $N \in NS(S)$. Then, (i) $Su^n(\overrightarrow{\psi^n}(N)) \geq \overrightarrow{\psi^n}(Su^n(N))$, and (ii) $So^n(\overrightarrow{\psi^n}(N)) \geq \overrightarrow{\psi^n}(So^n(N))$.*

Proof: We prove only (ii). For $s_3 \in R$,
$F_{\overrightarrow{\psi}(So(N))}(s_3) = \vee\{F_{So(N)}(s_1) : s_1 \in S, \psi(s_1) = s_3\}$
$= \vee\{\vee\{F_N(s_2) \otimes F_\delta(s_1, u, s_2) : s_2 \in S, u \in X\} : s_1 \in S, \psi(s_1) = s_3\}$
$= \vee\{F_N(s_2) \otimes F_\delta(s_1, u, s_2) : s_2 \in S, u \in X\}$.
Similarly, $G_{\overrightarrow{\psi}(So(N))}(s_3) = \wedge\{G_N(s_2) \otimes G_\delta(s_1, u, s_2) : s_2 \in S, u \in X\}$, and $H_{\overrightarrow{\psi}(So(N))}(s_3) = \wedge\{H_N(s_2) \otimes H_\delta(s_1, u, s_2) : s_2 \in S, u \in X\}$. Now, We have
$F_{So(\overrightarrow{\psi}(N))}(s_3) = \vee\{F_{\overrightarrow{\psi}(N)}(s_4) \otimes F_\lambda(s_3, u, s_4) : s_4 \in R, u \in X\}$
$= \vee\{\vee\{F_N(s_2) \otimes F_\lambda(s_3, u, s_4) : \psi(s_2) = s_4, s_2 \in S\} : s_4 \in R, u \in X\}$
$= \vee\{F_N(s_2) \otimes F_\lambda(\psi(s_1), u, \psi(s_2)) : s_2 \in S, u \in X\}$
$\geq \vee\{F_N(s_2) \otimes F_\delta(s_1, u, s_2) : s_2 \in S, u \in X\} = F_{\overrightarrow{\psi}(So(N))}(s_3)$.
Similarly, $G_{So(\overrightarrow{\psi}(N))}(s_3) \leq G_{\overrightarrow{\psi}(So(N))}(s_3)$, and $H_{So(\overrightarrow{\psi}(N))}(s_3) \leq H_{\overrightarrow{\psi}(So(N))}(s_3)$.
Hence, $So^n(\overrightarrow{\psi^n}(N)) \geq \overrightarrow{\psi^n}(So^n(N))$.

Proposition 5.6 *Let $\psi : A \to A'$ be a NHOM between two NAut $A = (S, X, \delta^n)$ and $A' = (R, X, \lambda^n)$ with $N \in NS(S)$. Then, $\mu^n(\overrightarrow{\psi}^n(N)) \geq \overrightarrow{\psi}^n(\mu^n(N))$.*

Proof: For $s_3 \in R$, $F_{\overrightarrow{\psi}(\mu(N))}(s_3) = \vee\{F_{\mu(N)}(s_1) : s_1 \in S, \psi(s_1) = s_3\} = \vee\{\wedge\{F_\delta(s_2, u, s_1) \to F_N(s_2) : s_2 \in S, u \in X\} : s_1 \in S, \psi(s_1) = s_3\}; G_{\overrightarrow{\psi}(\mu(N))}(s_3) = \wedge\{G_{\mu(N)}(s_1) : s_1 \in S, \psi(s_1) = s_3\} = \wedge\{\vee\{G_\delta(s_2, u, s_1) \leftarrow G_N(s_2) : s_2 \in S, u \in X\} : s_1 \in S, \psi(s_1) = s_3\}$. Similarly, $H_{\overrightarrow{\psi}(\mu(N))}(s_3) = \wedge\{\vee\{H_\delta(s_2, u, s_1) \leftarrow H_N(s_2) : s_2 \in S, u \in X\} : s_1 \in S, \psi(s_1) = s_3\}$. Now we have,

$F_{\mu(\overrightarrow{\psi}(N))}(s_3) = \wedge\{F_\lambda(s_4, u, s_3) \to F_{\overrightarrow{\psi}(N)}(s_4) : s_4 \in R, u \in X\} \geq \wedge\{\vee\{F_\lambda(s_4, u, s_3) \to F_N(s_2) : s_2 \in S, \psi(s_2) = s_4\} : s_4 \in R, u \in X\} = \wedge\{\vee\{F_\lambda(\psi(s_2), u, \psi(s_1)) \to F_N(s_2) : s_2 \in S\} : u \in X\} \geq \wedge\{\vee\{F_\delta(s_2, u, s_1) \to F_N(s_2) : s_2 \in S, u \in X\}\} \geq F_{\overrightarrow{\psi}(\mu(N))}(s_3); G_{\mu(\overrightarrow{\psi}(N))}(s_3) = \vee\{G_\lambda(s_4, u, s_3) \leftarrow G_{\overrightarrow{\psi}(N)}(s_4) : s_4 \in R, u \in X\} \leq \vee\{\wedge\{G_\lambda(s_4, u, s_3) \leftarrow G_N(s_2) : s_2 \in S, \psi(s_2) = s_4\} : s_4 \in R, u \in X\} = \vee\{\wedge\{G_\lambda(\psi(s_2), u, \psi(s_1)) \leftarrow G_N(s_2) :$

$s_2 \in S\} : u \in X\} \leq \vee\{\wedge\{G_\delta(s_2, u, s_1) \leftarrow G_N(s_2) : s_2 \in S, u \in X\}\} \leq G_{\overrightarrow{\psi}(\mu(N))}(s_3)$. Similarly, $H_{\mu(\overrightarrow{\psi}(N))}(s_3) \leq H_{\overrightarrow{\psi}(\mu(N))}(s_3)$. Hence, $\mu^n(\overrightarrow{\psi^n}(N)) \geq \overrightarrow{\psi^n}(\mu^n(N))$.

Remark 5.1 *The Propositions 5.5 and 5.6 also hold for SNHOM.*

6 Conclusions

In this study, we have explored the algebraic and topological characteristics of neutrosophic automata by utilizing various operators, such as the neutrosophic source, neutrosophic successor, and neutrosophic core operators. It has been demonstrated that these operators serve as a foundation for applying neutrosophic topological and co-topological concepts. In conclusion, we have highlighted that these operators exhibit notable properties when considered under neutrosophic homomorphism.

7 Future Research

In future, we will try to study the categorical properties of neutrosophic co-topology/neutrosophic topology and neutrosophic topology on neutrosophic rough sets.

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Chapter 16

Applications of a Few Operations on Pythagorean Neutrosophic Super Hypersoft Matrices

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ABSTRACT

The Pythagorean Neutrosophic Super Hypersoft Matrix (PNSHSM) Theory is the concept under investigation in this chapter. Here, we offer several new notations (operations) such PNSHS-sub matrix, Equal PNSHSM, Null PNSHSM, Universal PNSHSM, and Complement PNSHSM. PNSHSM is the matrix representation of a Pythagorean Neutrosophic Super Hypersoft Set (PNSHSS).n recent times, refined neutrosophic sets introduced by Deli [6] has been one of the most powerful and flexible approaches for dealing with complex and uncertain situations of real world.

Keywords: Neutrosophic Soft Set, Neutrosophic Hypersoft set, Pythagorean Neutrosophic Hypersoft set, Pythagorean Neutrosophic Super Hypersoft set, Pythagorean Neutrosphic Super Hyper soft Matrix.

1. INTRODUCTION

A. Zadeh established a foundation for fuzzy sets in 1965 [14]. The level of membership values determines the fuzzy sets. In certain instances, assigning membership values to fuzzy sets may be challenging. To address the uncertainty around membership values, Turksen suggested the idea of interval-valued fuzzy sets [13]. For accurate representation of an item in a dubious and uncertain state, we need to take membership and non-membership values in particular real-world problems. Atanassaov [2] first developed IFS, which are useful in certain situations. Insufficient data can be handled using IFS, which takes into account both truth and falsehood values.

Smarandache was the first to propose the idea of the Neutrosophic set [10]. The neutrosophic set indicates the membership values for truth, uncertainty, and falsehood. Molodstov [7] introduced the concept of a soft set as a brand-new numerical tool for handling problems with ambiguous situations. According to him, a family of universal sets with parameter subsets is a soft set. Soft sets are helpful in many areas, such as artificial intelligence, game theory, and simple decision-making issues [5]. Over the past few years, various researchers have examined the fundamentals of soft set theory. A theoretical analysis of soft sets was by Maji et al.[6] addresses the subset and super set were presented by Ali at el.[1]. Smarandache suggested a fresh approach for dealing with uncertainty. He expanded the $\text{Softset}_{(SS)}$ to $\text{Hypersoft}_{(HS)}$ set and Super Hyper soft set related to the Smarandache power set [12]. Then Jayasudha and Raghavi introduced the concept of Neutrosophic Hypersoft Matrices and their applications in 2024. Hemalatha and Francina Shalini defined the concept of the Pythagorean Neutrosophic Super Hypersoft set. This is closer to our everyday life.

2. PRELIMINARIES

Definition 1. [6] $\mathfrak{U}$, $\text{P}(\mathfrak{U})$ represents the universe set and power set respectively, Z be parameters. Consider $\text{Å} \subset Z$. Then $(F, \text{Å})$ is a soft set over $\mathfrak{U}$, where $F: \text{Å} \longrightarrow \text{P}(\mathfrak{U})$.

Definition 2 [5] $\mathfrak{U}$ −universal set and ω -set of attributes with respect to $\mathfrak{U}$. Let $\text{P}(\mathfrak{U})$-set of Neutrosophic values of $\mathfrak{U}$ and $\text{Å} \subseteq \omega$. A pair $(F, \text{Å})$is called a Neutrosophic soft set over $\mathfrak{U}$ and $F: \text{Å} \longrightarrow \text{P}(\mathfrak{U})$.

Definition 3. [11] U represent the universe and the power set of U is $\mathfrak{W}(U)$. For t ≥ 1 let $(\mathfrak{S}_1, \mathfrak{S}_2, \ldots \mathfrak{S}_n)_{HS}$ be t-distinct attributes, each of whose associated attributive values is the set $(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3 \ldots \mathfrak{s}_{t)_{HS}}$ with $(\mathfrak{s}_y \cap \mathfrak{s}_z)_{HS} = \emptyset$ as well as $y \neq z$ and $y, z \in \{1,2,, \ldots \ldots, t\}$, Then $(\mathfrak{F}, \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3 \ldots \mathfrak{s}_t)_{HS}$ is a Hypersoft set over U.

where $\mathfrak{F}$: $(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3 \ldots \mathfrak{s}_t)_{HS} \longrightarrow \mathfrak{W}$ (U)

Definition 4. [8] $\mathfrak{U}$ -universal set, P($\mathfrak{U}$)- be a power set of $\mathfrak{U}$. Let $H_1, H_2, H_3 \ldots H_v$ for v$\geq$ 1 be n well -defined attributes, whose corresponding attributive values are respectively the set $h_1, h_2, h_3 \ldots h_v$ with $h_i \cap h_j = \emptyset$ for i≠j and i,j $\in \{1,2,3, \ldots \ldots, v\}$ and their relation $h_1, h_2, h_3 \ldots = \mathcal{H}$,then the pair (F, $\mathcal{H}$) is said to be Neutrosophic Hypersoft set (PNHSS in short) over $\mathfrak{U}$ where F: $h_1, h_2, h_3 \ldots h_v \longrightarrow$ P($\mathfrak{U}$)

and (F,$h_1, h_2, h_3 \ldots h_v$)={($\mathcal{H}$,<x,$T_{F(\mathcal{H})}(x)$, $I_{F(\mathcal{H})}(x)$, $F_{F(\mathcal{H})}(x)$>) : $x \in \mathfrak{U}$,}where T is the membership value of truthiness, I is the membership value of indeterminacy and F is the membership value of falsity such that $T_{F(\mathcal{H})}(x)$, $I_{F(\mathcal{H})}(x)$, $F_{F(\mathcal{H})}(x) \in [0,1]$ also $0 \leq T_{F(\mathcal{H})}(x), I_{F(\mathcal{H})}(x), F_{F(\mathcal{H})}(x) \leq 3$.

Definition 5. [9] $\mathfrak{U}$ -universal set and P($\mathfrak{U}$) -power set. Consider $H_1, H_2, H_3 \ldots H_v$ for v$\geq$ 1 be n well -defined attributes, whose corresponding attributive values are respectively the set $h_1, h_2, h_3 \ldots h_v$ with $h_i \cap h_j = \emptyset$ for i≠j and i,j $\in \{1,2,3, \ldots \ldots, v\}$ and their relation $h_1, h_2, h_3 \ldots h_v = \mathcal{H}$,then the pair (F, $\mathcal{H}$)is said to be Pythagorean Neutrosophic Hypersoft set (PNHSS in short) over U where F: $h_1, h_2, h_3 \ldots h_v \longrightarrow$ P($\mathfrak{U}$)

and $(F, h_1, h_2, h_3 \ldots h_v)_{PNHSS}$={($\mathcal{H}$,<x,$T_{F(\mathcal{H})}(x)$, $I_{F(\mathcal{H})}(x)$, $F_{F(\mathcal{H})}(x)$>) : $x \in \mathfrak{U}$ }where the membership values for truthfulness, indeterminacy, and falsehood are denoted by T, I, and F, respectively, such that $T_{F(\mathcal{H})}(x)$, $I_{F(\mathcal{H})}(x)$, $F_{F(\mathcal{H})}(x) \in [0,1]$ also $0 \leq T_{F(\mathcal{H})}(x)^2 + I_{F(\mathcal{H})}(x)^2 + F_{F(\mathcal{H})}(x)^2 \leq 2$.

Definition 6.[4] Let U $= \{u^1, u^2, \ldots u^\alpha\}$ and (U) - universal set and power set of the universal set and also, Consider $L_1, L_2, L_3 \ldots L_\beta$ for $\beta \geq 1$ be β well -defined attributes, whose corresponding attributive values are respectively the set $l_1^a, l_2^b, l_3^c \ldots l_\beta^z$ and their relation set $L_1^a \times L_2^b \times L_3^c \times \ldots \times L_\beta^z$ were $a, b, c, \ldots z = 1,2, .. \beta$ then the pair $(G, L_1^a \times L_2^b \times L_3^c \times \ldots \times L_\beta^z)$ is said to be Neutrosophic Hypersoft set over U where G: $L_1^a \times L_2^b \times L_3^c \times \ldots \times L_\beta^z \longrightarrow \mathrm{P}(U)$, and it defined as,

$G\ (L_1^a \times L_2^b \times L_3^c \times \ldots \times L_\beta^z) = \{u, \eta_i(u), \nu_i(u), \varphi_i(u) : u \in U, i \in L_1^a \times L_2^b \times L_3^c \times \ldots \times L_\beta^z\}$

Let $R_i = L_1^a \times L_2^b \times L_3^c \times \ldots \times L_\beta^z$ be the relationship, and its defining role is

$x_{R_i} = L_1^a \times L_2^b \times L_3^c \times \ldots \times L_\beta^z \longrightarrow \mathrm{P}(U)$;

It is defined as $X_{R_i} = \{u, \eta_i(u), \nu_i(u), \varphi_i(u) : u \in U, i \in L_1^a \times L_2^b \times L_3^c \times \ldots \times L_\beta^z\}$

and can be a representation of R_i as given in Table 1.

U	L_1^a	L_2^b	⋯	L_β^z
u^1	$x_{R_i}(u^1, \mathrm{L}_1^a)$	$x_{R_i}(u^1, \mathrm{L}_2^b)$	⋯	$x_{R_i}(u^1, \mathrm{L}_\beta^z)$
u^2	$x_{R_i}(u^2, \mathrm{L}_1^a)$	$x_{R_i}(u^2, \mathrm{L}_2^b)$	⋯	$x_{R_i}(u^2, \mathrm{L}_\beta^z)$
⋮	⋮	⋮	⋮	⋮
u^α	$x_{R_i}(u^\alpha, \mathrm{L}_1^a)$	$x_{R_i}(u^\alpha, \mathrm{L}_2^b)$	⋯	$x_{R_i}(u^\alpha, \mathrm{L}_\beta^z)$

If $M_{ij} = x_{R_i}(u^i, \mathrm{L}_j^k)$

Where $i = 1,2, \cdots \alpha$, and $k = a, b, \cdots z$, $j = 1,2, \cdots n$

Then the matrix is defined as, $\left[M_{ij}\right]_{\alpha\times\beta} = \begin{pmatrix} m_{11} & m_{12} & \dots & m_{1\beta} \\ m_{21} & m_{22} & \dots & m_{2\beta} \\ \vdots & \vdots & \vdots & \vdots \\ m_{\alpha 1} & m_{\alpha 2} & \dots & m_{\alpha\beta} \end{pmatrix}$

Where,

$$M_{ij} = (\eta_{\mathrm{L}_j^k}(u_i), \nu_{\mathrm{L}_j^k}(u_i), \varphi_{\mathrm{L}_j^k}(u_i) : u_i \in U, \mathrm{L}_j^k \in L_1^a \times L_2^b \times L_3^c \times \dots \times L_\beta^z) = (\eta_{ij}^M, \nu_{ij}^M . \varphi_{ij}^M)$$

Operations of NHSM: Let $S = [s_{st}] = (T_{stn}^s, I_{stn}^s, F_{stn}^s)$ and $\mathbb{P} = [p_{st}] = \left(T_{stn}^p, I_{stn}^p, F_{stn}^p\right)$ be two NHSMs with order $\partial \times \mathcal{E}$,

- **Arithmetic mean $S \otimes \mathbb{P} = 0$**

Where $T_{stn}^o = \frac{T_{stn}^s + T_{stn}^p}{2}, I_{stn}^o = \frac{I_{stn}^s + I_{stn}^p}{2}, F_{stn}^o = \frac{F_{stn}^s + F_{stn}^p}{2} \quad \forall\, s, t, n.$

- **Weighted Arithmetic mean $S \otimes^w \mathbb{P} = 0$**

$$T_{stn}^o = \frac{w_1 T_{stn}^s + w_2 T_{stn}^s}{w_1 + w_2}, I_{stn}^o = \frac{w_1 I_{stn}^s + w_2 I_{stn}^s}{w_1 + w_2},$$

$$F_{stn}^o = \frac{w_1 F_{stn}^s + w_2 F_{stn}^s}{w_1 + w_2} \quad \forall s, t, n \text{ and } w_1, w_2 > 0$$

- **Geometric mean $S \boxtimes \mathbb{P} = 0$**

Where $T_{stn}^o = \sqrt{T_{stn}^s . T_{stn}^p, I_{stn}^s} = \sqrt{I_{stn}^s . I_{stn}^p, F_{stn}^s} = \sqrt{F_{stn}^s . F_{stn}^p}, \forall\, s, t, n.$

- **Weighted Geometric Mean $S \boxtimes^w \mathbb{P} = 0$**

$$T_{stn}^o = (w_1 + w_2)\sqrt{(T_{stn}^s)^{w_1} . (T_{stn}^s)^{w_2}},$$

$$I_{stn}^o = (w_1 + w_2)\sqrt{(I_{stn}^s)^{w_1} . (I_{stn}^s)^{w_2}},$$

$$F_{stn}^o = (w_1 + w_2)\sqrt{(F_{stn}^s)^{w_1} . (F_{stn}^s)^{w_2}}$$

$$\forall s, t, n \text{ and } w_1, w_2 > 0$$

- **Harmonic mean $S \blacksquare \mathbb{P} = 0$**

$$T_{stn}^o = \frac{2T_{stn}^s . T_{stn}^p}{T_{stn}^s + T_{stn}^p}, I_{stn}^o = \frac{2I_{stn}^s . I_{stn}^p}{I_{stn}^s + I_{stn}^p}, F_{stn}^o = \frac{2F_{stn}^s . F_{stn}^p}{F_{stn}^s + F_{stn}^p} \quad \forall\, s, t, n.$$

- **Weighted Harmonic mean $S \blacksquare^w \mathbb{P} = 0$**

Where $T_{stn}^o = \frac{w_1 + w_2}{\frac{w_1}{T_{stn}^s} + \frac{w_2}{T_{stn}^s}}, I_{stn}^o = \frac{w_1 + w_2}{\frac{w_1}{I_{stn}^s} + \frac{w_2}{I_{stn}^s}}, F_{stn}^o = \frac{w_1 + w_2}{\frac{w_1}{F_{stn}^s} + \frac{w_2}{F_{stn}^s}},$

$$\forall s, t, n \text{ and } w_1, w_2 > 0$$

Definition 7.[3] $\mathfrak{V}$, P($\mathfrak{V}$) – be the universal set and power set. Consider $H_1, H_2, \dots . H_v$ for v≥ 1 be n-distinct attributes, whose corresponding attributive values are respectively the set $h_1, h_2, \dots . h_v$ with $h_m \cap h_n = \emptyset$ for m≠n and m,n ∈ {1,2,, … … , v}. Let P(h_1) , P(h_2) , … … , P(h_v) $= \mathcal{H}$ be the power sets of the set $h_1, h_2, h_3 \dots . h_v$ respectively.

Then the pair $\left(F,\ \mathrm{P}(h_1) \times \mathrm{P}(h_2) \times \mathrm{P}(h_3) \times, \dots \dots , \times \mathrm{P}(h_v)\right)$ is said to be PNSHSS over $\mathfrak{V}$.

where F: P(h_1) × P(h_2) × P(h_3) ×, … … ,× P(h_n) ⟶ P($\mathfrak{V}$) and

$$F\left(\mathrm{P}(h_1)\ \times\ \mathrm{P}(h_2)\times \mathrm{P}(h_3)\ \times,\dots\dots.,\times \mathrm{P}(h_v)\right)_{PNSHSS} = \left\{ \begin{matrix} \mathcal{H}, < x, T_{F(\mathcal{H})}(x), I_{F(\mathcal{H})}(x), F_{F(\mathcal{H})}(x) >:\ x \in \mathfrak{V} \\ \mathcal{H} \in \mathrm{P}(h_1)\ \times\ \mathrm{P}(h_2)\times \mathrm{P}(h_3)\ \times,\dots\dots.,\times \mathrm{P}(h_n) \end{matrix} \right\}$$

Where $T_{F(\mathcal{H})}$ and $F_{F(\mathcal{H})}$ are the dependent components. $I_{F(\mathcal{H})}$ is independent component. Also,

$0\leq \left(T_{F(\mathcal{H})}(x)\right)^2 + \left(I_{F(\mathcal{H})}(x)\right)^2 + \left(F_{F(\mathcal{H})}(x)\right)^2 \leq 2$ and $T_{F(\mathcal{H})}(x)\ + F_{F(\mathcal{H})}(x)\ \leq 1$.

3. Pythagorean Neutrosophic Super Hypersoft Matrices

Definition 8 $\mathfrak{V} = u^1, u^2, \dots u^\gamma$ be the universal set and P($\mathfrak{V}$) - power set of $\mathfrak{V}$. Consider $\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3 \dots\dots \mathfrak{G}_\alpha$ for $\alpha \geq 1$ be α -distinct attributes, whose corresponding attributive values are respectively the set $\mathfrak{G}_1^a, \mathfrak{G}_2^b, \mathfrak{G}_3^c \dots\dots \mathfrak{G}_\alpha^z$ with $\mathfrak{G}_\mathfrak{r} \cap \mathfrak{G}_\mathfrak{s} = \emptyset$ for $\mathfrak{r} \neq \mathfrak{s}$ and $\mathfrak{r},\mathfrak{s} \in \{1,2,,\dots\dots,\alpha\}$. Let $\mathrm{P}(\mathfrak{G}_1^a)$, $\mathrm{P}(\mathfrak{G}_2^b), \mathrm{P}(\mathfrak{G}_3^c)$, $\dots\dots, \mathrm{P}(\mathfrak{G}_\alpha^z) = \mathcal{H}$ be the power sets of the set $\mathfrak{G}_1^a, \mathfrak{G}_2^b, \mathfrak{G}_3^c \dots\dots \mathfrak{G}_\alpha^z$ respectively.

Then the pair $\left(F,\ \mathrm{P}(\mathfrak{G}_1^a)\ \times\ \mathrm{P}(\mathfrak{G}_2^b)\times \mathrm{P}(\mathfrak{G}_3^c)\ \times,\dots.,\times \mathrm{P}(\mathfrak{G}_\alpha^z)\right)$ is said to be PNSHSS over $\mathfrak{V}$.

where $\mathfrak{F}$: $\mathrm{P}(\mathfrak{G}_1^a)\ \times\ \mathrm{P}(\mathfrak{G}_2^b)\times \mathrm{P}(\mathfrak{G}_3^c)\ \times,\dots\dots.,\times \mathrm{P}(\mathfrak{G}_\alpha^z) \longrightarrow \mathrm{P}(\mathfrak{V})$ and

$$\mathfrak{F}\left(\mathrm{P}(\mathfrak{G}_1^a)\ \times\ \mathrm{P}(\mathfrak{G}_2^b)\times \mathrm{P}(\mathfrak{G}_3^c)\ \times,\dots\dots.,\times \mathrm{P}(\mathfrak{G}_\alpha^z)\right) = \left\{ \begin{matrix} \mathcal{H}, < x, T_{\mathfrak{F}(\mathcal{H})}(x), I_{\mathfrak{F}(\mathcal{H})}(x), F_{\mathfrak{F}(\mathcal{H})}(x) >:\ x \in \mathfrak{V}, \\ \mathcal{H} \in \mathrm{P}(\mathfrak{G}_1^a)\ \times\ \mathrm{P}(\mathfrak{G}_2^b)\ \times,\dots\dots.,\times \mathrm{P}(\mathfrak{G}_\alpha^z) \end{matrix} \right\}$$

Where $T_{F(\mathcal{H})}$ and $F_{F(\mathcal{H})}$ are the dependent components. $I_{F(\mathcal{H})}$ is independent component. Also,

$0\leq \left(T_{F(\mathcal{H})}(x)\right)^2 + \left(I_{F(\mathcal{H})}(x)\right)^2 + \left(F_{F(\mathcal{H})}(x)\right)^2 \leq 2$ and $T_{F(\mathcal{H})}(x)\ + F_{F(\mathcal{H})}(x)\ \leq 1$.

Let $H_i =\ \mathrm{P}(\mathfrak{G}_1^a)\ \times\ \mathrm{P}(\mathfrak{G}_2^b)\ \times,\dots\dots.,\times \mathrm{P}(\mathfrak{G}_\alpha^z)$

$x_{\mathfrak{H}_\mathfrak{i}} =\ \mathrm{P}(\mathfrak{G}_1^a)\ \times\ \mathrm{P}(\mathfrak{G}_2^b)\ \times,\dots\dots.,\times \mathrm{P}(\mathfrak{G}_\alpha^z)$

It is defined as $x_{\mathfrak{H}_\mathfrak{i}} = \left\{x, T_{\mathfrak{F}(H)}(x), I_{\mathfrak{F}(H)}(x), F_{\mathfrak{F}(H)}(x)\right\} x \in \mathfrak{V}, \mathfrak{i} \in \mathrm{P}(\mathfrak{G}_1^a)\ \times\ \mathrm{P}(\mathfrak{G}_2^b)\ \times,\dots\dots.,\times \mathrm{P}(\mathfrak{G}_\alpha^z)$ and can be representation of $\mathfrak{H}_\mathfrak{i}$ as given in the table

U	$\mathrm{P}(\mathfrak{G}_1^a)$	$\mathrm{P}(\mathfrak{G}_2^b)$	⋯	$\mathrm{P}(\mathfrak{G}_\alpha^z)$
u^1	$x_{\mathfrak{H}_\mathfrak{i}}\left(u^1, \mathrm{P}(\mathfrak{G}_1^a)\right)$	$x_{\mathfrak{H}_\mathfrak{i}}\left(u^1, \mathrm{P}(\mathfrak{G}_2^b)\right)$	⋯	$x_{\mathfrak{H}_\mathfrak{i}}\left(u^1, \mathrm{P}(\mathfrak{G}_\alpha^z)\right)$
u^2	$x_{\mathfrak{H}_\mathfrak{i}}\left(u^2, \mathrm{P}(\mathfrak{G}_1^a)\right)$	$x_{\mathfrak{H}_\mathfrak{i}}\left(u^2, \mathrm{P}(\mathfrak{G}_2^b)\right)$	⋯	$x_{\mathfrak{H}_\mathfrak{i}}\left(u^2, \mathrm{P}(\mathfrak{G}_\alpha^z)\right)$
⋮	⋮	⋮	⋮	⋮
u^δ	$x_{\mathfrak{H}_\mathfrak{i}}\left(u^\delta, \mathrm{P}(\mathfrak{G}_1^a)\right)$	$x_{\mathfrak{H}_\mathfrak{i}}\left(u^\delta, \mathrm{P}(\mathfrak{G}_2^b)\right)$	⋯	$x_{\mathfrak{H}_\mathfrak{i}}\left(u^\delta, \mathrm{P}(\mathfrak{G}_\alpha^z)\right)$

If $M_{\mathfrak{s}\mathfrak{t}} = x_{\mathfrak{H}_\mathfrak{i}}\left(u^\ell, \mathrm{P}(\mathfrak{G}_m^n)\right)$

Where $\ell = 1,2,\cdots\delta,\ n = a, b, \cdots z,\quad m = 1,2,\cdots\alpha$

Then the matrix is defined as,

$$[M_{\mathfrak{s}\mathfrak{t}}]_{\propto\times n} = \begin{pmatrix} \mathfrak{m}_{11} & \mathfrak{m}_{12} & \dots & \mathfrak{m}_{1n} \\ \mathfrak{m}_{21} & \mathfrak{m}_{22} & \dots & \mathfrak{m}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \mathfrak{m}_{\propto 1} & \mathfrak{m}_{\propto 2} & \dots & \mathfrak{m}_{\propto n} \end{pmatrix}$$

Where $M_{\mathfrak{s}\mathfrak{t}} = \left(T_{\mathrm{P}(\mathfrak{G}_m^n)}(u_\mathfrak{i}), I_{\mathrm{P}(\mathfrak{G}_m^n)}(u_\mathfrak{i}), F_{\mathrm{P}(\mathfrak{G}_m^n)}(u_\mathfrak{i}), u_\mathfrak{i} \in U, \mathrm{P}(\mathfrak{G}_m^n) \in\ \mathrm{P}(\mathfrak{G}_1^a)\ \times\ \mathrm{P}(\mathfrak{G}_2^b)\times \mathrm{P}(\mathfrak{G}_3^c)\ \times,\dots\dots.,\times \mathrm{P}(\mathfrak{G}_\alpha^z)\ \right) = (T_{\mathfrak{s}\mathfrak{t}n}^M, \mathrm{I}_{\mathfrak{s}\mathfrak{t}n}^M, F_{\mathfrak{s}\mathfrak{t}n}^M)$

By using the Pythagorean Neutrosophic Super Hypersoft matrix (PNSHSM), we may therefore express any Pythagorean Neutrosophic Super Hypersoft Set.

Example 9: Let γ be the collection of mobile phones within the showroom:

$\gamma = \{Lava, Tecno, Poco, iTel\}$

The decision maker offers his conclusion around the choice procedure of the alternatives such as;

$\gamma = \{Lava, Tecno, Poco, iTel\}$

$\mathfrak{G}_1 = \{Storage\} = \mathfrak{G}_1^a = \{64GB, 128GB, 256GB\}$

$\mathfrak{G}_2 = \{Camera\ Resolution\} = \mathfrak{G}_2^b = \{20Mp, 50Mp\}$

$\mathfrak{G}_3 = \{Battery\} = \mathfrak{G}_3^c = \{4200mAh, 4800mAh\}$

$\mathfrak{G}_4 = \{RAM\} = \mathfrak{G}_4^d = \{6GB, 8GB, 12GB\}$

$$\mathrm{P}(\mathfrak{G}_1^a) = \begin{Bmatrix} \{64GB\}, \{128GB\}, \{256GB\}, \{64GB, 128GB\}\{64GB, 256GB\}, \\ \{128GB, 256GB\}, \{64GB, 128GB, 256GB\}, \emptyset \end{Bmatrix}$$

$$\mathrm{P}(\mathfrak{G}_2^b) = \{\{20Mp\}, \{50Mp\}, \{20Mp, 50Mp\}, \emptyset\}$$

$$\mathrm{P}(\mathfrak{G}_3^c) = \{\{4200\ mAh\}, \{4800\ mAh\}, \{4200\ mAh, 4800\ mAh\}, \emptyset\}$$

$$\mathrm{P}(\mathfrak{G}_4^d) = \{\{6GB\}, \{8GB\}, \{12GB\}, \{6GB, 8\ GB\}, \{6GB, 12GB\}, \{8GB, 12\ GB\}, \{6GB, 8GB, 12\ GB\}, \emptyset\}$$

Let $S: \mathrm{P}(\mathfrak{G}_1^a) \times \mathrm{P}(\mathfrak{G}_2^b) \times, \ldots\ldots, \times \mathrm{P}(\mathfrak{G}_\alpha^z) \to P(\gamma)$

Pythagorean Neutrosophic Super Hypersoft set is characterized as:

$$S: \mathrm{P}(\mathfrak{G}_1^a) \times \mathrm{P}(\mathfrak{G}_2^b) \times \mathrm{P}(\mathfrak{G}_3^c) \times, \ldots\ldots, \times \mathrm{P}(\mathfrak{G}_\alpha^z) \to P(\gamma)$$

Table 1: Pythagorean Neutrosophic Super Hypersoft Values

	Lava	Tecno	Poco	iTel
64GB	(0.2, 0.7, 0.1)	(0.3, 0.1, 0.5)	(0.5, 0.3, 0.5)	(0.6, 0.2, 0.1)
128GB	(0.3, 0.5, 0.2)	(0.7, 0.1, 0.2)	(0.6, 0.2, 0.3)	(0.8, 0.5, 0.1)
256 GB	(0.4, 0.3, 0.6)	(0.6, 0.5, 0.4)	(0.7, 0.1, 0.2)	(0.5, 0.2, 0.3)
{64 GB, 128 GB}	(0.3, 0.5, 0.6)	(0.6, 0.5, 0.4)	(0.6, 0.2, 0.3)	(0.5, 0.2, 0.3)
{64 GB, 256 GB}	(0.3, 0.5, 0.2)	(0.6, 0.1, 0.4)	(0.6, 0.2, 0.3)	(0.6, 0.5, 0.1)
{128 GB, 256 GB}	(0.2, 0.7, 0.2)	(0.3, 0.1, 0.5)	(0.5, 0.3, 0.5)	(0.6, 0.2, 0.1)
{64 GB, 128 GB, 256 GB}	(0.4, 0.3, 0.1)	(0.7, 0.1, 0.2)	(0.7, 0.1, 0.2)	(0.8, 0.2, 0.1)
$\emptyset$	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)
20 Mp	(0.3, 0.6, 0.6)	(0.2, 0.4, 0.6)	(0.4, 0.5, 0.4)	(0.5, 0.2, 0.2)
50 Mp	(0.1, 0.7, 0.4)	(0.6, 0.5, 0.2)	(0.3, 0.1, 0.2)	(0.8, 0.3, 0.1)
{20 Mp, 50 Mp}	(0.3, 0.6, 0.4)	(0.6, 0.4, 0.2)	(0.4, 0.1, 0.2)	(0.8, 0.2, 0.1)

∅	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)
4200 mAh	(0.6, 0.1, 0.2)	(0.3, 0.2, 0.6)	(0.1, 0.7, 0.5)	(0.3, 0.4, 0.5)
4800 mAh	(0.7, 0.2, 0.3)	(0.5, 0.3, 0.4)	(0.6, 0.1, 0.4)	(0.5, 0.4, 0.3)
{4200 mAh, 4800 mAh}	(0.7, 0.1, 0.2)	(0.5, 0.2, 0.4)	(0.6, 0.1, 0.4)	(0.5, 0.4, 0.3)
∅	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)
6GB	(0.6, 0.5, 0.3)	(0.5, 0.5, 0.5)	(0.9, 0.1, 0.1)	(0.6, 0.7, 0.1)
8 GB	(0.8, 0.4, 0.1)	(0.5, 0.2, 0.5)	(0.8, 0.1, 0.2)	(0.1, 0.6, 0.7)
12 GB	(0.9, 0.1, 0.1)	(0.8, 0.5, 0.1)	(0.6, 0.5, 0.3)	(0.3, 0.5, 0.2)
{6 GB, 8 GB}	(0.8, 0.4, 0.1)	(0.5, 0.5, 0.5)	(0.6, 0.1, 0.3)	(0.3, 0.6, 0.2)
{6 GB, 12 GB}	(0.8, 0.4, 0.1)	(0.5, 0.5, 0.5)	(0.8, 0.1, 0.2)	(0.1, 0.6, 0.7)
{8 GB, 12 GB}	(0.6, 0.5, 0.3)	(0.5, 0.5, 0.5)	(0.8, 0.1, 0.2)	(0.3, 0.7, 0.2)
{6GB, 8GB, 12 GB}	(0.9, 0.1, 0.1)	(0.8, 0.2, 0.1)	(0.9, 0.1, 0.1)	(0.6, 0.5, 0.1)
∅	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)

Let us assume $\mathfrak{F}$: $\mathrm{P}(\mathfrak{G}_1^a) \times \mathrm{P}(\mathfrak{G}_2^b) \times \mathrm{P}(\mathfrak{G}_3^c) \times, \dots\dots, \times \mathrm{P}(\mathfrak{G}_\alpha^z) =$
$S(256\ GB, \{20\ Mp, 150\ Mp\}, \{4200\ mAh, 4800\ mAh\}, 8GB) = \{Techno, Poco\}$

Then, the Pythagorean Neutrosophic Super Hypersoft of the above expected connection is:

$$S: \mathrm{P}(\mathfrak{G}_1^a) \times \mathrm{P}(\mathfrak{G}_2^b) \times \mathrm{P}(\mathfrak{G}_3^c) \times, \dots\dots, \times \mathrm{P}(\mathfrak{G}_\alpha^z) = S(256\ GB, \{20Mp, 50Mp\}, \{4200\ mAh, 4800\ mAh\}, 8GB)$$

$$= \begin{Bmatrix} \big(Tecno, (0.6, 0.5, 0.4), (0.6, 0.4, 0.2), (0.5, 0.2, 0.4), (0.5, 0.2, 0.5\ )\big), \\ \big(Poco, (0.6, 0.1, 0.4), (0.4, 0.1, 0.2), (0.6, 0.1, 0.4), (0.8, 0.1, 0.2)\big) \end{Bmatrix}$$

Additionally, it may be shown as follows in matrix form:

$$[S]_{2\times 4} = \begin{pmatrix} (0.6, 0.5, 0.4) & (0.6, 0.4, 0.2) & (0.5, 0.2, 0.4) & (0.5, 0.2, 0.5) \\ (0.7, 0.1, 0.2) & (0.4, 0.1, 0.2) & (0.6, 0.1, 0.4) & (0.8, 0.1, 0.2) \end{pmatrix}$$

Definition 10 Let $S = [s_{st}]$ and $\mathbb{P} = [p_{st}]$ be two Pythagorean Neutrosophic Super Hypersoft matrix Set with order $\partial \times \mathfrak{E}$, where $S_{st} = (T_{stn}^s, I_{stn}^s, F_{stn}^s)$ and $P_{st} = \big(T_{stn}^p, I_{stn}^p, F_{stn}^p\big)$. S is said to Pythagorean Neutrosophic Super Hypersoft Sub matrix of P if $T_{stn}^s \leq T_{stn}^p, I_{stn}^s \leq I_{stn}^p, F_{stn}^s \geq F_{stn}^p$.

Example 11: Consider PNSHS Matrix $[S]_{2\times 4}$ of example 9.

$$[S]_{2\times 4} = \begin{pmatrix} (0.6, 0.5, 0.4) & (0.6, 0.4, 0.2) & (0.5, 0.2, 0.4) & (0.5, 0.2, 0.5) \\ (0.7, 0.1, 0.2) & (0.4, 0.1, 0.2) & (0.6, 0.1, 0.4) & (0.8, 0.1, 0.2) \end{pmatrix}$$

Consider another PNSHS matrix $[\mathbb{P}]$ related with the Pythagorean Neutrosophic Super Hypersoft Set.

$$\mathbb{P}: \mathrm{P}(\mathfrak{G}_1^a) \times \mathrm{P}(\mathfrak{G}_2^b) \times \mathrm{P}(\mathfrak{G}_3^c) \times, \dots\dots, \times \mathrm{P}(\mathfrak{G}_\alpha^z) \to P(\gamma)$$

Over the same universe and attributes as in Example 9.

$$\mathrm{P}(\mathfrak{G}_1^a) \times \mathrm{P}(\mathfrak{G}_2^b) \times, \dots\dots, \times \mathrm{P}(\mathfrak{G}_\alpha^z) = \mathbb{P}(256\ GB, \{20\ Mp, 50\ Mp\}, \{4200\ mAh, 4800\ mAh\}, 8GB)$$

$$= \begin{Bmatrix} \big(Tecno, (0.7, 0.6, 0.3), (0.8, 0.5, 0.1), (0.7, 0.3, 0.3), (0.8, 0.5, 0.1)\big), \\ \big(Pocco, (0.8, 0.5, 0.1), (0.6, 0.3, 0.1), (0.7, 0.5, 0.1), (0.9, 0.3, 0.2)\big) \end{Bmatrix}$$

Hence the PNSHS matrix $[P]$ is written as,

$$[\mathbb{P}]_{2x4} = \begin{pmatrix} (0.7,0.6,0.3) & (0.8,0.5,0.1) & (0.7,0.3,0.3) & (0.8,0.5,0,1) \\ (0.8,0.5,0.1) & (0.6,0.3,0.2) & (0.7,0.5,0.1) & (0.9,0.3,0.2) \end{pmatrix}$$

The Pythagorean Neutrosophic Super Hypersoft Sub matrix condition is thus satisfied by the membership values of "Tecno" and "Poco" for 256GB in both sets, which are (0.7,0.6,0.3) and (0.8,0.5,0.1), respectively. This demonstrates that (0.7,0.6,0.3) ⊆ (0.8,0.5,0.1), and the remaining $[S]\ amd\ PNSHSM[\mathbb{P}]$ qualities were similarly affected.

Definition12 Take $[S] = [s_{st}]$ and $[\mathbb{P}] = [p_{st}]$ be two Pythagorean Neutrosophic Super Hyper soft Matrix with order $\partial \times \mathcal{E}$, where $s_{st} = (T^s_{stn}, I^s_{stn}, F^s_{stn})$ and $p_{st} = \left(T^p_{stn}, I^p_{stn}, F^p_{stn}\right)$. Then S is said to be the Equal PNSHSM of $\mathbb{P}$ if,

$T^s_{stn} = T^p_{stn},\ I^s_{stn} = I^p_{stn},\quad F^s_{stn} = F^p_{stn}$

Example13: Consider PNSHS Matrix $[S]_{2\times4}$ of example 9.

$$[S]_{2\times4} = \begin{pmatrix} (0.6, 0.5, 0.4) & (0.6, 0.4, 0.2) & (0.5, 0.2, 0.4) & (0.5, 0.2, 0.5) \\ (0.7, 0.1, 0.2) & (0.4, 0.1, 0.2) & (0.6, 0.1, 0.4) & (0.8, 0.1, 0.2) \end{pmatrix}$$

Let's look at another PNSHSM $[\mathbb{P}]$ that is associated with the Pythagorean Neutrosophic Super Hypersoft set $\mathbb{P}: \mathrm{P}(\mathfrak{G}_1^a) \times \mathrm{P}(\mathfrak{G}_2^b) \times, \ldots\ldots, \times \mathrm{P}(\mathfrak{G}_\alpha^z) \to P(\gamma)$ over the same universe and characteristics as seen in Example 1.

$$\left(\mathrm{P}(\mathfrak{G}_1^a) \times \mathrm{P}(\mathfrak{G}_2^b) \times, \ldots\ldots, \times \mathrm{P}(\mathfrak{G}_\alpha^z)\right) = \mathbb{P}(256\ GB, \{20\ Mp, 50\ Mp\}, \{4200\ mAh, 4800\ mAh\}, 8GB)$$
$$= \begin{Bmatrix} (Tecno, (0.6, 0.5, 0.4), (0.6, 0.4, 0.2), (0.5, 0.2, 0.4), (0.5, 0.2, 0.5)), \\ (Pocco, (0.7, 0.1, 0.2), (0.4, 0.1, 0.2), (0.6, 0.1, 0.4), (0.8, 0.1, 0.2)) \end{Bmatrix}$$

Then PNSHSM $[\mathbb{P}]$ is written as,

$$[\mathbb{P}]_{2\times4} = \begin{pmatrix} (0.6, 0.5, 0.4) & (0.6, 0.4, 0.2) & (0.5, 0.2, 0.4) & (0.5, 0.2, 0.5) \\ (0.7, 0.1, 0.2) & (0.4, 0.1, 0.2) & (0.6, 0.1, 0.4) & (0.8, 0.1, 0.2) \end{pmatrix}$$

Consequently, we can see that the membership values of "Tecno" and "Pocco" for 256GB in both matrices meet the Equal PNSHSM criteria, which are 0.6=0.6, 0.5=0.5, 0.4=0.4, and 0.7=0.7, 0.1=0.1, 0.2=0.2. As de monstrated by this, (0.6,0.5,0.4) = (0.6,0.5,0.4) and (0.7,0.1,0.2) = (0.7,0.1,0.2), and the same was true for the remaining PNSHSM characteristics $[S]\ and\ [\mathbb{P}]$.

Definition 14 Let $S = [s_{st}]$ be Pythagorean Neutrosophic Super Hypersoft matrix Set with order $\partial \times \mathcal{E}$, where $s_{st} = (T^s_{stn}, I^s_{stn}, F^s_{stn})$. Then S- Null Pythagorean Neutrosophic Super Hypersoft matrix if

$$T^s_{stn} = 0, I^s_{stn} = 1, F^s_{stn} = 1.$$

Example 15: A Null-PNSHSM S is then provided by taking the same universe and characteristics as in Example 9.

$$[S]_{2\times4} = \begin{pmatrix} (0.0,1.0,1.0) & (0.0,1.0,1.0) & (0.0,1.0,1.0) & (0.0,1.0,1.0) \\ (0.0,1.0,1.0) & (0.0,1.0,1.0) & (0.0,1.0,1.0) & (0.0,1.0,1.0) \end{pmatrix}$$

Definition 16 Let $S = [s_{st}]$ be Pythagorean Neutrosophic Super Hypersoft matrix Set with order $\partial \times \mathcal{E}$, where $s_{st} = \left(T^s_{ijk}, I^s_{ijk}, F^s_{ijk}\right)$. Consequently, if matrix S is a Universal Pythagorean Neutrosophic Super Hypersoft matrix, then

$$T^s_{ijk} = 1, I^s_{ijk}0, F^s_{ijk} = 0.$$

Example 17: A Universal-PNSHSM S is then provided by taking the same universe and characteristics as in Example 9.

$$[S]_{2\times4} = \begin{pmatrix} (1.0,0.0,0.0) & (1.0,0.0,0.0) & (1.0,0.0,0.0) & (1.0,0.0,0.0) \\ (1.0,0.0,0.0) & (1.0,0.0,0.0) & (1.0,0.0,0.0) & (1.0,0.0,0.0) \end{pmatrix}$$

Definition 18 Let $S = [s_{st}]$ be Pythagorean Neutrosophic Super Hypersoft matrix Set with order $\partial \times \mathcal{E}$, where $s_{st} = (T^s_{stn}, I^s_{stn}, F^s_{stn})$. Then $S = [s_{st}]^{\ c}$ the matrix S is said to be Complement Pythagorean Neutrosophic Super Hypersoft matrix of $S = [s_{st}]$ if

$$(T^s_{stn}, I^s_{stn}, F^s_{stn})^c = (F^s_{stn}, I^s_{stn}, T^s_{stn})$$

Example19: Consider PNSHS Matrix $[S]_{2\times4}$ of example 9.

$$[S]_{2\times4} = \begin{pmatrix} (0.6, 0.5, 0.4) & (0.6, 0.4, 0.2) & (0.5, 0.2, 0.4) & (0.5, 0.2, 0.5) \\ (0.7, 0.1, 0.2) & (0.4, 0.1, 0.2) & (0.6, 0.1, 0.4) & (0.8, 0.1, 0.2) \end{pmatrix}$$

Then

$$[S]^{c}{}_{2\times4} = \begin{pmatrix} (0.4, 0.5, 0.6) & (0.2, 0.4, 0.6) & (0.4, 0.2, 0.5) & (0.5, 0.2, 0.5) \\ (0.2, 0.1, 0.7) & (0.2, 0.1, 0.4) & (0.4, 0.1, 0.6) & (0.2, 0.1, 0.8) \end{pmatrix}$$

4. Properties of Pythagorean Neutrosophiic Super Hypersoft Matrices

Proposition 20 Let $S_{\mathcal{Y}} = [s_{st}]$, $\mathbb{P}_{\mathcal{Y}} = [p_{st}]$ and $O_{\mathcal{Y}} = [o_{st}]$ where $s_{st} = (T^s_{stn}, I^s_{stn}, F^s_{stn}), p_{st} = \left(T^p_{stn}, I^p_{stn}, F^p_{stn}\right)$ *and* $o_{st} = (T^o_{stn}, I^o_{stn}, F^o_{stn}) \in$ PNSHSMs with order $\partial \times \mathcal{E}$. Then,

i. $S_{\mathcal{Y}} \otimes (\mathbb{P}_{\mathcal{Y}} \cup O_{\mathcal{Y}}) = (S_{\mathcal{Y}} \otimes \mathbb{P}_{\mathcal{Y}}) \cup (S_{\mathcal{Y}} \otimes O_{\mathcal{Y}})$, $(S_{\mathcal{Y}} \cup \mathbb{P}_{\mathcal{Y}}) \otimes O_{\mathcal{Y}} = (S_{\mathcal{Y}} \otimes \mathbb{P}_{\mathcal{Y}}) \cup (S_{\mathcal{Y}} \otimes O_{\mathcal{Y}})$

ii. $S_{\mathcal{Y}} \otimes (\mathbb{P}_{\mathcal{Y}} \cap O_{\mathcal{Y}}) = (S_{\mathcal{Y}} \otimes \mathbb{P}_{\mathcal{Y}}) \cap (S_{\mathcal{Y}} \otimes O_{\mathcal{Y}})$, $(S_{\mathcal{Y}} \cap \mathbb{P}_{\mathcal{Y}}) \otimes O_{\mathcal{Y}} = (S_{\mathcal{Y}} \otimes \mathbb{P}_{\mathcal{Y}}) \cap (S_{\mathcal{Y}} \otimes O_{\mathcal{Y}})$

iii. $S_{\mathcal{Y}} \boxtimes (\mathbb{P}_{\mathcal{Y}} \cup O_{\mathcal{Y}}) \neq (S_{\mathcal{Y}} \boxtimes \mathbb{P}_{\mathcal{Y}}) \cup (S_{\mathcal{Y}} \boxtimes O_{\mathcal{Y}})$, $(S_{\mathcal{Y}} \cup \mathbb{P}_{\mathcal{Y}}) \boxtimes O_{\mathcal{Y}} \neq (S_{\mathcal{Y}} \boxtimes \mathbb{P}_{\mathcal{Y}}) \cup (S_{\mathcal{Y}} \boxtimes O_{\mathcal{Y}})$

iv. $S_{\mathcal{Y}} \boxtimes (\mathbb{P}_{\mathcal{Y}} \cap O_{\mathcal{Y}}) \neq (S_{\mathcal{Y}} \boxtimes \mathbb{P}_{\mathcal{Y}}) \cap (S_{\mathcal{Y}} \boxtimes O_{\mathcal{Y}})$, $(S_{\mathcal{Y}} \cap \mathbb{P}_{\mathcal{Y}}) \boxtimes O_{\mathcal{Y}} \neq (S_{\mathcal{Y}} \boxtimes \mathbb{P}_{\mathcal{Y}}) \cap (S_{\mathcal{Y}} \boxtimes O_{\mathcal{Y}})$

v. $S_{\mathcal{Y}} \blacksquare (\mathbb{P}_{\mathcal{Y}} \cup O_{\mathcal{Y}}) \neq (S_{\mathcal{Y}} \blacksquare \mathbb{P}_{\mathcal{Y}}) \cup (S_{\mathcal{Y}} \blacksquare O_{\mathcal{Y}})$, $(S_{\mathcal{Y}} \cup \mathbb{P}_{\mathcal{Y}}) \blacksquare O_{\mathcal{Y}} \neq (S_{\mathcal{Y}} \blacksquare \mathbb{P}_{\mathcal{Y}}) \cup (S_{\mathcal{Y}} \blacksquare O_{\mathcal{Y}})$

vi. $S_{\mathcal{Y}} \blacksquare (\mathbb{P}_{\mathcal{Y}} \cap O_{\mathcal{Y}}) \neq (S_{\mathcal{Y}} \blacksquare \mathbb{P}_{\mathcal{Y}}) \cap (S_{\mathcal{Y}} \blacksquare O_{\mathcal{Y}})$, $(S_{\mathcal{Y}} \cap \mathbb{P}_{\mathcal{Y}}) \blacksquare O_{\mathcal{Y}} \neq (S_{\mathcal{Y}} \blacksquare \mathbb{P}_{\mathcal{Y}}) \cap (S_{\mathcal{Y}} \blacksquare O_{\mathcal{Y}})$

Proof

i. $S_{\mathcal{Y}} \otimes (\mathbb{P}_{\mathcal{Y}} \cup O_{\mathcal{Y}})$, $(S_{\mathcal{Y}} \otimes \mathbb{P}_{\mathcal{Y}}) \cup (S_{\mathcal{Y}} \otimes O_{\mathcal{Y}}) \in PNSHSM$ *with order* $\partial \times \mathcal{E}$. Then

$$S_{\mathcal{Y}} \otimes (\mathbb{P}_{\mathcal{Y}} \cup O_{\mathcal{Y}}) = [(T^s_{stn}, I^s_{stn}, F^s_{stn})] \otimes \left[\left(\max(T^p_{stn}, T^o_{stn}), \frac{(I^p_{stn}, I^o_{stn})}{2}, min(F^p_{stn}, F^o_{stn})\right)\right]$$

$$= \left[\left(\frac{T^s_{stn} + max(T^p_{stn}, T^o_{stn})}{2}, \frac{I^s_{stn} + \frac{(I^p_{stn}, I^o_{stn})}{2}}{2}, \frac{F^s_{stn} + min(F^p_{stn}, F^o_{stn})}{2}\right)\right]$$

$$= \left[\left(max\left[\left(\frac{(T^s_{stn}, I^p_{stn})}{2}\right), \left(\frac{(I^s_{stn}, I^o_{stn})}{2}\right)\right], \frac{\frac{(I^s_{stn}, I^p_{ijk})}{2} + \frac{(I^s_{stn}, I^o_{stn})}{2}}{2}, min\left[\left(\frac{(F^s_{stn}, F^p_{stn})}{2}\right), \left(\frac{(F^s_{stn}, F^o_{stn})}{2}\right)\right]\right)\right]$$

$$= \left[\left(\frac{(T^s_{stn}, T^p_{stn})}{2}, \frac{(I^s_{stn}, I^p_{stn})}{2}, \frac{(F^s_{stn}, F^p_{stn})}{2}\right)\right] \cup \left[\left(\frac{(T^s_{stn}, T^o_{stn})}{2}, \frac{(I^s_{stn}, I^o_{stn})}{2}, \frac{(F^s_{stn}, F^o_{stn})}{2}\right)\right]$$

$$S_{\mathcal{Y}} \otimes (\mathbb{P}_{\mathcal{Y}} \cup O_{\mathcal{Y}}) = (S_{\mathcal{Y}} \otimes \mathbb{P}_{\mathcal{Y}}) \cup (S_{\mathcal{Y}} \otimes O_{\mathcal{Y}})$$

ii. $S_{\mathcal{Y}} \otimes (\mathbb{P}_{\mathcal{Y}} \cap O_{\mathcal{Y}}), (S_{\mathcal{Y}} \otimes \mathbb{P}_{\mathcal{Y}}) \cap (S_{\mathcal{Y}} \otimes O_{\mathcal{Y}}) \in PNSHSM$ *with order* $\partial \times \mathcal{E}$. Then

$$S_{\mathcal{Y}} \otimes (\mathbb{P}_{\mathcal{Y}} \cap O_{\mathcal{Y}}) = [(T^s_{stn}, I^s_{stn}, F^s_{stn})] \otimes \left[\left(\min(T^p_{stn}, T^o_{stn}), \frac{(I^p_{stn}, I^o_{stn})}{2}, max(F^p_{stn}, F^o_{stn})\right)\right]$$

$$= \left[\left(\frac{T^s_{stn} + min(T^p_{stn}, T^o_{stn})}{2}, \frac{I^s_{stn} + \frac{(I^p_{stn}, I^o_{stn})}{2}}{2}, \frac{F^s_{stn} + max(F^p_{stn}, F^o_{stn})}{2}\right)\right]$$

$$= \left[\left(min\left[\left(\frac{(T^s_{stn}, I^p_{stn})}{2}\right), \left(\frac{(T^s_{stn}, T^o_{stn})}{2}\right)\right], \frac{\frac{(I^s_{stn}, I^p_{stn})}{2} + \frac{(I^s_{stn}, I^o_{stn})}{2}}{2}, max\left[\left(\frac{(F^s_{stn}, F^p_{stn})}{2}\right), \left(\frac{(F^s_{stn}, F^o_{stn})}{2}\right)\right]\right)\right]$$

$$= \left[\left(\frac{(T^s_{stn}, I^p_{stn})}{2}, \frac{(I^s_{stn}, I^p_{stn})}{2}, \frac{(F^s_{stn}, F^p_{stn})}{2}\right)\right] \cap \left[\left(\frac{(T^s_{stn}, T^o_{stn})}{2}, \frac{(I^s_{stn}, I^o_{stn})}{2}, \frac{(F^s_{stn}, F^o_{stn})}{2}\right)\right]$$

$$S_{\mathcal{Y}} \otimes (\mathbb{P}_{\mathcal{Y}} \cap O_{\mathcal{Y}}) = (S_{\mathcal{Y}} \otimes \mathbb{P}_{\mathcal{Y}}) \cap (S_{\mathcal{Y}} \otimes O_{\mathcal{Y}})$$

iii. Here $(S_{\mathcal{Y}} \cup \mathbb{P}_{\mathcal{Y}}) \boxtimes O_{\mathcal{Y}}, (S_{\mathcal{Y}} \boxtimes \mathbb{P}_{\mathcal{Y}}) \cup (S_{\mathcal{Y}} \boxtimes O_{\mathcal{Y}}) \in PNSHSM$ *with order* $\partial \times \mathcal{E}$. Then

$$(S_{\mathcal{Y}} \cup \mathbb{P}_{\mathcal{Y}}) \boxtimes O_{\mathcal{Y}} = \left[\left(\max(T^s_{stn}, T^p_{stn}), \frac{(I^s_{stn}, I^p_{stn})}{2}, min(F^s_{stn}, F^p_{stn})\right)\right] \boxtimes [(T^o_{stn}, I^o_{stn}, F^o_{stn})]$$

$$= \left[\left(\sqrt{\max(T^s_{stn}, T^p_{stn})}.T^o_{ijk}, \sqrt{\frac{(I^s_{stn}, I^p_{stn})}{2}}.I^o_{stn}, \sqrt{min(F^s_{stn}, F^p_{stn}).F^o_{stn}}\right)\right]$$

$$= \left\lfloor \begin{matrix} max\left(\sqrt{(T^s_{stn}, T^o_{stn})}, \sqrt{(T^p_{stn}, T^o_{stn})}\right), \left(\sqrt{\frac{(I^s_{stn}, I^o_{stn})}{2}} + \sqrt{\frac{(I^p_{stn}, I^o_{stn})}{2}}\right), \\ min\left(\sqrt{(T^s_{stn}, T^o_{stn})}, \sqrt{(T^p_{stn}, T^o_{stn})}\right) \end{matrix} \right\rfloor$$

$(S_{\mathcal{Y}} \boxtimes \mathbb{P}_{\mathcal{Y}}) \cup (S_{\mathcal{Y}} \boxtimes O_{\mathcal{Y}})$

$$= \left\lfloor \left\langle max\left(\sqrt{(T^s_{\dot{s}tn}, T^o_{\dot{s}tn})}, \sqrt{(T^p_{\dot{s}tn}, T^o_{\dot{s}tn})}\right), \left\langle \frac{\sqrt{(I^s_{\dot{s}tn}, I^o_{\dot{s}tn})} + \sqrt{(I^p_{\dot{s}tn}, I^o_{\dot{s}tn})}}{2} \right\rangle, min\left(\sqrt{(T^s_{\dot{s}tn}, T^o_{\dot{s}tn})}, \sqrt{(T^p_{\dot{s}tn}, T^o_{\dot{s}tn})}\right) \right\rangle \right\rfloor$$

Therefore, we have $(S_{\mathfrak{y}} \cup \mathbb{P}_{\mathfrak{y}}) \boxtimes O_{\mathfrak{y}} \neq (S_{\mathfrak{y}} \boxtimes \mathbb{P}_{\mathfrak{y}}) \cup (S_{\mathfrak{y}} \boxtimes O_{\mathfrak{y}})$.

iv. Here $(S_{\mathfrak{y}} \cap \mathbb{P}_{\mathfrak{y}}) \boxtimes O_{\mathfrak{y}}, (S_{\mathfrak{y}} \boxtimes \mathbb{P}_{\mathfrak{y}}) \cap (S_{\mathfrak{y}} \boxtimes O_{\mathfrak{y}}) \in PNSHSM\ with\ order\ \partial \times \mathcal{E}$. Then

$$(S_{\mathfrak{y}} \cap \mathbb{P}_{\mathfrak{y}}) \boxtimes O_{\mathfrak{y}} = \left[\left(\min(T^s_{\dot{s}tn}, T^p_{\dot{s}tn}), \frac{(I^s_{\dot{s}tn}, I^p_{\dot{s}tn})}{2}, max(F^s_{\dot{s}tn}, F^p_{\dot{s}tn})\right)\right] \boxtimes [(T^o_{\dot{s}tn}, I^o_{\dot{s}tn}, F^o_{\dot{s}tn})]$$

$$= \left[\left(\sqrt{\min(T^s_{\dot{s}tn}, T^p_{\dot{s}tn})}.T^o_{\dot{s}tn}, \sqrt{\frac{(I^s_{\dot{s}tn}, I^p_{\dot{s}tn})}{2}}.I^o_{\dot{s}tn}, \sqrt{max(F^s_{\dot{s}tn}, F^p_{\dot{s}tn}).F^o_{\dot{s}tn}}\right)\right]$$

$$= \left[\left(min\left(\sqrt{(T^s_{\dot{s}tn}, T^o_{\dot{s}tn})}, \sqrt{(T^p_{\dot{s}tn}, T^o_{\dot{s}tn})}\right), \left(\sqrt{\frac{(I^s_{\dot{s}tn}, I^o_{\dot{s}tn})}{2}} + \sqrt{\frac{(I^p_{\dot{s}tn}, I^o_{\dot{s}tn})}{2}}\right), max\left(\sqrt{(T^s_{\dot{s}tn}, T^o_{\dot{s}tn})}, \sqrt{(T^p_{\dot{s}tn}, T^o_{\dot{s}tn})}\right)\right)\right]$$

$(S \boxtimes \mathbb{P}) \cap (S \boxtimes O_{\mathfrak{y}})$

$$= \left\lfloor \left\langle min\left(\sqrt{(T^s_{\dot{s}tn}, T^o_{\dot{s}tn})}, \sqrt{(T^p_{\dot{s}tn}, T^o_{\dot{s}tn})}\right), \left\langle \frac{\sqrt{(I^s_{\dot{s}tn}, I^o_{\dot{s}tn})} + \sqrt{(I^p_{\dot{s}tn}, I^o_{\dot{s}tn})}}{2} \right\rangle, max\left(\sqrt{(T^s_{\dot{s}tn}, T^o_{\dot{s}tn})}, \sqrt{(T^p_{\dot{s}tn}, T^o_{\dot{s}tn})}\right) \right\rangle \right\rfloor$$

Therefore, we have $(S_{\mathfrak{y}} \cap \mathbb{P}_{\mathfrak{y}}) \boxtimes O_{\mathfrak{y}} \neq (S_{\mathfrak{y}} \boxtimes \mathbb{P}_{\mathfrak{y}}) \cap (S_{\mathfrak{y}} \boxtimes O_{\mathfrak{y}})$.

v. Here $(S_{\mathfrak{y}} \cup \mathbb{P}_{\mathfrak{y}}) \blacksquare O_{\mathfrak{y}}$, $(S_{\mathfrak{y}} \blacksquare \mathbb{P}_{\mathfrak{y}}) \cup (S_{\mathfrak{y}} \blacksquare O_{\mathfrak{y}}) \in PNSHSM\ with\ order\ \partial \times \mathcal{E}$. Then

$$(S_{\mathfrak{y}} \cup \mathbb{P}_{\mathfrak{y}}) \blacksquare O_{\mathfrak{y}} = \left[\left(\max(T^s_{\dot{s}tn}, T^p_{\dot{s}tn}), \frac{(I^s_{\dot{s}tn}, I^p_{\dot{s}tn})}{2}, min(F^s_{\dot{s}tn}, F^p_{\dot{s}tn})\right)\right] \blacksquare [(T^o_{\dot{s}tn}, I^o_{\dot{s}tn}, F^o_{\dot{s}tn})]$$

$$= \left[\left(\frac{\max(T^s_{\dot{s}tn}, T^p_{\dot{s}tn}).2T^o_{ijk}}{\max(T^s_{\dot{s}tn}, T^p_{\dot{s}tn}) + T^o_{\dot{s}tn}}, \frac{\frac{(I^s_{\dot{s}tn}, I^p_{\dot{s}tn})}{2}.2I^o_{ijk}}{\frac{(I^s_{\dot{s}tn}, I^p_{\dot{s}tn})}{2} + I^o_{\dot{s}tn}}, \frac{min(F^s_{\dot{s}tn}, F^p_{\dot{s}tn}).2F^o_{\dot{s}tn}}{min(F^s_{\dot{s}tn}, F^p_{\dot{s}tn}) + F^o_{\dot{s}tn}}\right)\right]$$

$(S_{\mathfrak{y}} \cup \mathbb{P}_{\mathfrak{y}}) \blacksquare O_{\mathfrak{y}}$

$$= \left[\left(max\left(\frac{2T^s_{\dot{s}tn}.T^o_{\dot{s}tn}}{T^s_{\dot{s}tn} + T^o_{\dot{s}tn}}, \frac{2T^p_{\dot{s}tn}.T^o_{\dot{s}tn}}{T^p_{\dot{s}tn} + T^o_{\dot{s}tn}}\right), \left(\frac{2I^s_{\dot{s}tn}.I^o_{\dot{s}tn}}{I^s_{\dot{s}tn} + I^o_{\dot{s}tn}}, \frac{2I^p_{\dot{s}tn}.T^o_{\dot{s}tn}}{I^p_{\dot{s}tn} + I^o_{\dot{s}tn}}\right), min\left(\frac{2F^s_{\dot{s}tn}.F^o_{\dot{s}tn}}{F^s_{\dot{s}tn} + F^o_{\dot{s}tn}}, \frac{2F^p_{\dot{s}tn}.F^o_{\dot{s}tn}}{F^p_{\dot{s}tn} + F^o_{\dot{s}tn}}\right)\right)\right]$$

$(S_{\mathfrak{y}} \blacksquare \mathbb{P}_{\mathfrak{y}}) \cup (S_{\mathfrak{y}} \blacksquare O_{\mathfrak{y}})$

$$= \left\lfloor \left\langle max\left(\frac{2T^s_{\dot{s}tn}.T^o_{\dot{s}tn}}{T^s_{\dot{s}tn} + T^o_{\dot{s}tn}}, \frac{2T^p_{\dot{s}tn}.T^o_{\dot{s}tn}}{T^p_{\dot{s}tn} + T^o_{\dot{s}tn}}\right), \frac{\left(\frac{2I^s_{\dot{s}tn}.I^o_{\dot{s}tn}}{I^s_{\dot{s}tn} + I^o_{\dot{s}tn}}, \frac{2I^p_{\dot{s}tn}.T^o_{\dot{s}tn}}{I^p_{\dot{s}tn} + I^o_{\dot{s}tn}}\right)}{2}, min\left(\frac{2F^s_{\dot{s}tn}.F^o_{\dot{s}tn}}{F^s_{\dot{s}tn} + F^o_{\dot{s}tn}}, \frac{2F^p_{\dot{s}tn}.F^o_{\dot{s}tn}}{F^p_{\dot{s}tn} + F^o_{\dot{s}tn}}\right) \right\rangle \right\rfloor$$

Therefore, we have, $(S_{\mathfrak{y}} \cup \mathbb{P}_{\mathfrak{y}}) \blacksquare O_{\mathfrak{y}} \neq (S_{\mathfrak{y}} \blacksquare \mathbb{P}_{\mathfrak{y}}) \cup (S_{\mathfrak{y}} \blacksquare O_{\mathfrak{y}})$

vi. Here $(S_{\mathfrak{y}} \cap \mathbb{P}_{\mathfrak{y}}) \blacksquare O_{\mathfrak{y}}$, $(S_{\mathfrak{y}} \blacksquare \mathbb{P}_{\mathfrak{y}}) \cap (S_{\mathfrak{y}} \blacksquare O_{\mathfrak{y}}) \in PNSHSM\ with\ order\ \partial \times \mathcal{E}$. Then

$$(S_{\mathfrak{y}} \cap \mathbb{P}_{\mathfrak{y}}) \blacksquare O_{\mathfrak{y}} = \left[\left(\min(T^s_{\dot{s}tn}, T^p_{\dot{s}tn}), \frac{(I^s_{\dot{s}tn}, I^p_{\dot{s}tn})}{2}, max(F^s_{\dot{s}tn}, F^p_{\dot{s}tn})\right)\right] \blacksquare [(T^o_{\dot{s}tn}, I^o_{\dot{s}tn}, F^o_{\dot{s}tn})]$$

$$= \left[\left(\frac{\min(T^s_{\dot{s}tn}, T^p_{\dot{s}tn}).2T^o_{\dot{s}tn}}{\min(T^s_{\dot{s}tn}, T^p_{\dot{s}tn}) + T^o_{\dot{s}tn}}, \frac{\frac{(I^s_{\dot{s}tn}, I^p_{\dot{s}tn})}{2}.2I^o_{\dot{s}tn}}{\frac{(I^s_{\dot{s}tn}, I^p_{\dot{s}tn})}{2} + I^o_{\dot{s}tn}}, \frac{max(F^s_{\dot{s}tn}, F^p_{\dot{s}tn}).2F^o_{\dot{s}tn}}{max(F^s_{\dot{s}tn}, F^p_{\dot{s}tn}) + F^o_{\dot{s}tn}}\right)\right]$$

$(S_{\mathfrak{y}} \cap \mathbb{P}_{\mathfrak{y}}) \blacksquare O_{\mathfrak{y}}$

$$= \left[\left(min\left(\frac{2T^s_{\dot{s}tn}.T^o_{\dot{s}tn}}{T^s_{\dot{s}tn} + T^o_{\dot{s}tn}}, \frac{2T^p_{\dot{s}tn}.T^o_{\dot{s}tn}}{T^p_{\dot{s}tn} + T^o_{\dot{s}tn}}\right), \left(\frac{2I^s_{\dot{s}tn}.I^o_{\dot{s}tn}}{I^s_{\dot{s}tn} + I^o_{\dot{s}tn}}, \frac{2I^p_{\dot{s}tn}.I^o_{\dot{s}tn}}{I^p_{\dot{s}tn} + I^o_{\dot{s}tn}}\right), max\left(\frac{2F^s_{\dot{s}tn}.F^o_{\dot{s}tn}}{F^s_{\dot{s}tn} + F^o_{\dot{s}tn}}, \frac{2F^p_{\dot{s}tn}.F^o_{\dot{s}tn}}{F^p_{\dot{s}tn} + F^o_{\dot{s}tn}}\right)\right)\right]$$

$$(S_{\mathfrak{y}} \blacksquare \mathbb{P}_{\mathfrak{y}}) \cap (S_{\mathfrak{y}} \blacksquare O_{\mathfrak{y}})$$

$$= \left\lfloor \left\langle min\left(\frac{2T^s_{\dot{s}tn}.T^o_{\dot{s}tn}}{T^s_{\dot{s}tn}+T^o_{\dot{s}tn}}, \frac{2T^p_{\dot{s}tn}.T^o_{\dot{s}tn}}{T^p_{\dot{s}tn}+T^o_{\dot{s}tn}}\right), \frac{\left(\frac{2I^s_{\dot{s}tn}.I^o_{\dot{s}tn}}{I^s_{\dot{s}tn}+I^o_{\dot{s}tn}}, \frac{2I^p_{\dot{s}tn}.I^o_{\dot{s}tn}}{I^p_{\dot{s}tn}+I^o_{\dot{s}tn}}\right)}{2}, max\left(\frac{2F^s_{\dot{s}tn}.F^o_{\dot{s}tn}}{F^s_{\dot{s}tn}+F^o_{\dot{s}tn}}, \frac{2F^p_{\dot{s}tn}.F^o_{\dot{s}tn}}{F^p_{\dot{s}tn}+F^o_{\dot{s}tn}}\right) \right\rangle \right\rfloor$$

Therefore, we have, $(S_{\mathfrak{y}} \cap \mathbb{P}_{\mathfrak{y}}) \blacksquare O_{\mathfrak{y}} \neq (S_{\mathfrak{y}} \blacksquare \mathbb{P}_{\mathfrak{y}}) \cap (S_{\mathfrak{y}} \blacksquare O_{\mathfrak{y}})$

Proposition 21 Let $S_{\mathfrak{y}} = [s_{\dot{s}t}]$, where $s_{\dot{s}t} = (T^s_{\dot{s}tn}, I^s_{\dot{s}tn}, F^s_{\dot{s}tn}) \in PNSHSM\ with\ order\ \partial \times \mathcal{E}$. Then

i. $S_{\mathfrak{y}} \otimes^{w} S_{\mathfrak{y}} = S_{\mathfrak{y}}$
ii. $S_{\mathfrak{y}} \boxtimes^{w} S_{\mathfrak{y}} = S_{\mathfrak{y}}$
iii. $S_{\mathfrak{y}} \blacksquare^{w} S_{\mathfrak{y}} = S_{\mathfrak{y}}$

Proof For all $\mathcal{s}, t, n$ and $w_1, w_2 > 0$ we have

i. $$S_{\mathfrak{y}} \otimes^{w} S_{\mathfrak{y}} = \left[\left(\frac{w_1 T^s_{\dot{s}tn} + w_2 T^s_{\dot{s}tn}}{w_1+w_2}, \frac{w_1 I^s_{\dot{s}tn} + w_2 I^s_{\dot{s}tn}}{w_1+w_2}, \frac{w_1 F^s_{\dot{s}tn} + w_2 F^s_{\dot{s}tn}}{w_1+w_2}\right)\right] = [(T^s_{\dot{s}tn}, I^s_{\dot{s}tn}, F^s_{\dot{s}tn})] = S_{\mathfrak{y}}$$

ii. $$S_{\mathfrak{y}} \boxtimes^{w} S_{\mathfrak{y}} = \left[\begin{pmatrix} (w_1+w_2)\sqrt{(T^s_{\dot{s}tn})^{w_1}.(T^s_{\dot{s}tn})^{w_2}}, \\ (w_1+w_2)\sqrt{(I^s_{\dot{s}tn})^{w_1}.(I^s_{\dot{s}tn})^{w_2}}, \\ (w_1+w_2)\sqrt{(F^s_{\dot{s}tn})^{w_1}.(F^s_{\dot{s}tn})^{w_2}} \end{pmatrix}\right] = [(T^s_{\dot{s}tn}, I^s_{\dot{s}tn}, F^s_{\dot{s}tn})] = S_{\mathfrak{y}}$$

iii. $$S_{\mathfrak{y}} \blacksquare^{w} S_{\mathfrak{y}} = \left[\left(\frac{w_1+w_2}{\frac{w_1}{T^s_{\dot{s}tn}}+\frac{w_2}{T^s_{\dot{s}tn}}}, \frac{w_1+w_2}{\frac{w_1}{I^s_{\dot{s}tn}}+\frac{w_2}{I^s_{\dot{s}tn}}}, \frac{w_1+w_2}{\frac{w_1}{F^s_{\dot{s}tn}}+\frac{w_2}{F^s_{\dot{s}tn}}}\right)\right] = [(T^s_{\dot{s}tn}, I^s_{\dot{s}tn}, F^s_{\dot{s}tn})] = S_{\mathfrak{y}}$$

5. Conclusions

This chapter presents the PNSHS-sub matrix, Equal PNSHSM, Null PNSHSM, Universal PNSHSM, and Complement PNSHSM and few other operations of PNSHSMs. By providing a relevant example, the suggested operations and definitions' applicability and execution are confirmed. The Pythagorean Neutrosophic Super Hyper Soft Set, or PNSHSS, is a novel tool for appropriate selection in decision-making situations.

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Chapter 17

Topological Properties of Neutrosophic HyperEdge-Weighted SuperHyperGraphs and their applications in Social Networks

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Abstract. This study explores the topological properties of Neutrosophic HyperEdge-Weighted SuperHyperGraphs (NHEWSHGs) and their applications in social networks. These mathematical entities, called NHEWSHGs, consist of nodes and hyperedges with neutrosophic weights that analyze complex network structures with uncertain and partial truth components. The research defines essential topological properties through technical definitions and proves mathematical statements related to connectivity, cycle formation, and clustering, and provides examples to demonstrate their behavior. In this chapter, we introduce an algorithm which enables the conversion of data into NHEWSHGs along with practical implementation on a research collaboration network. NHEWSHGs show flexible application in studying real systems that contain uncertain relationships through their successful modeling of such systems, leading to valuable discoveries about network structures and dynamics.

Keywords: Neutrosophic; SuperHyperStructure; Neutrosophic SuperHyperStructure; SuperHyperGraphs; Neutrosophic SuperHyperGraphs.

1. Introduction

The interpretation of complex network relations struggles to handle uncertain data, ambiguous information, and incomplete truth while modeling real-world interaction methods in network theory. The efficiency of traditional graph models at unveiling network links does not include mechanisms to represent the uncertain elements commonly found in complex systems. The modeling and analysis of interconnected systems with uncertain conditions benefit from implementing NHEWSHGs as a knowledgeable framework. The theory of NHEWSHGs builds upon traditional hypergraph techniques by adding weight measurement mechanisms that use neutrosophic techniques. The extended framework enables users to assign truth, indeterminacy, and falsity elements to each hyperedge for advanced representation of network transformations. Smarandache introduced the concept of neutrosophy in his works ([1], [2], [3], [4]) to create a system for managing information characterized by indeterminate, inconsistent, and incomplete data frequently occurring in real applications. The theory expands classical and fuzzy logic [5] by separating the independent formulation of truth values and indeterminacy and falsity degrees.

Smarandache ([1], [2], [3], [4]) and other scholars extensively researched neutrosophic logic and its extensions while making neutrosophy a unifying field in logic. Through his work on neutrosophic logic, Smarandache ([6], [7]) introduced n-Valued Refined Neutrosophic Logic while creating score, certainty, and accuracy evaluation functions to determine total order of neutrosophic triplets, which have enhanced application potential in multiple fields ([8], [9], [10], [11], [12]). Das et al. ([13], [14]) assessed Gomati River water quality in Tripura using FSSs and neutrosophic approaches. Research-based implementation of advanced mathematical models

produced new MCDM techniques that enhance the efficiency of ecological decision processes. These methods, like VIKOR [15], TOPSIS (Dey et al., 2016) [16], TODIM [17], etc., to enhance accuracy on decision-making processes have been successful. Shil et al. [18] presented a hyperbolic cosine similarity measure for multi-attribute decision-making strategies in similarity measures and ranking in single-valued neutrosophic decision-making. Likewise, Ye et al. [19] applied single-valued neutrosophic multisets in medical diagnosis to enhance diagnostic accuracy using distance-based similarity measures. The neutrosophic framework has been extended over time to various mathematical structures: hypergraphs [20], algebraic systems [21], as well as to more complex structures like SuperHyperStructures [22]. On the other hand, Soft sets were extended to HyperSoft sets, then to Plithogenic Hypersoft sets [23], which greatly contributed to modeling uncertainty and indeterminate information. It is an evolution hidden due to systems' increasing complexity and the need for more sophisticated tools to handle them. This work deals with the generalization, called the neutrosophic sets, of different fuzzy set [5] theories, such as intuitionistic [24] and Pythagorean fuzzy set, and helps describe the complex decision-making problems [4].

The extension of the neutrosophic theory with SuperHyperFunction and Neutrosophic SuperHyperFunction enhances the implementation of neutrosophic theory in mathematical modeling and computation [25]. Das et al. [26] also investigated algebraic structures with single-valued pentapartitioned neutrosophic sets over which the theoretical basis of the neutrosophic system evolves. As in the above, we propose the concept of Neutrosophic d-Filter of d-Algebra to develop the understanding of neutrosophic algebraic structures [27]. They introduce a new vision in mathematical modeling, tending to more complicated aspects of uncertainty by introducing SuperHyperSoft sets and, more particularly, their fuzzy extensions [11]. These new trends in neutrosophic theory and applications were more objectively described by Smarandache and Pramanik ([16], [17]) in edited volumes; an overview of neutrosophic has offered a good review of theoretical and practical implications. Finally, the contributions made by Indian researchers in multi-attribute decision making under a neutrosophic environment have portrayed the flexibility and the effectiveness of neutrosophic sets in the complex decision-making processes [28]. It is also noted from the state-of-the-art review about neutrosophic sets and theory put forward by Otay and Kahraman [29] and the bibliometric analysis on neutrosophic set theory by Peng and Dai [30], that there is a growing interest and advancement in this field. In recent years, the neutrosophic set has been applied in the realms of communication, management, and information technology [31], and its integration with the single-valued neutrosophic set in the transdisciplinary approach [32]. Besides, the operations on single-valued neutrosophic graphs [24] and the construction of neutrosophic soft graphs [33] have furthered neutrosophic set application in the domains of graph theory. Works like" Beyond Neutrosophic Graphs" by Garrett [34] document the advancements in neutrosophic graph theory by exploring the complexities and possible applications of neutrosophic graphs as well as co-degree and degree of classes of neutrosophic hypergraphs. Other such examples include directing the early steps towards a theory of intuitionistic fuzzy graphs [35] as well as the extension of hypergraphs to n-SuperHyperGraphs [36] in the context of graph theory within the neutrosophic set framework.

The theoretical foundation of this study lies in the evolution of uncertainty modeling frameworks, beginning with fuzzy set (FS) theory introduced by Zadeh [5] and soft set (SS) theory proposed by Molodtsov [37]. Their fusion into fuzzy soft sets (FSSs) by Maji et al. (2001) provided a flexible tool for handling ambiguity, vagueness, and incomplete information in complex multi-criteria decision-making (MCDM) environments. Over the years, numerous extensions have broadened the applicability of these models. Intuitionistic fuzzy sets [38], neutrosophic sets ([1], [2]), and fuzzy soft multisets (FSMSs) have significantly enhanced decision-making frameworks under indeterminacy, making them particularly effective in domains as diverse as healthcare technologies, infrastructure design, environmental assessments, and socio-economic systems.

Recent contributions by Das and Granados have strengthened the adaptability of FSS-based MCDM models. They introduced FP-intuitionistic frameworks [39]. They extended these into FP-intuitionistic multi-fuzzy neutrosophic soft sets ([40], [41]), effectively treating vague, conflicting, and incomplete information more effectively. Earlier, Das ([42], [42]) developed weighted FSMSs and preference-based fuzzy rough models, which refined the ability to incorporate uncertainty and partial preferences. Their subsequent work proposed a group MCDM model using FSSs and weighted averages [44], and more recently, preference intuitionistic fuzzy rough models [45], highlighting nuanced inclusion properties in uncertain settings.

Parallel advancements include the development of weighted hesitant bipolar-valued FSSs ([46], [47]), applied effectively to urban river pollution assessment ([48], [49], [50]), thereby validating their utility in environmental decision-making. Extensions into inventory management in retail [51] and the introduction of an advanced weighted hypersoft expert system ([52], [53]) further broadened the scope of FSS applications in complex group decision contexts.

Complementary to these, Mukherjee and Das made notable contributions through the application of interval-valued intuitionistic FSMSs to real-life problems ([54], [55]), the development of Einstein operations for FSSs [56], and a comprehensive monograph on FSMS theory and applications [57]. Their works on topological and relational properties of FSSs and rough sets ([58], [59], [60]) expanded the theoretical landscape. Meanwhile, Granados et al. ([61], [62]) proposed advanced models involving continuous neutrosophic distributions and weighted neutrosophic soft multisets, strengthening MCDM strategies under high uncertainty.

Very recently, Das et al. [51] explored neutrosophic SuperHyper BCI-semigroups, establishing their algebraic relevance, while Das et al. ([52], [53]) advanced FSS-based MCDM models for adaptive inventory optimization and developed weighted hypersoft expert system frameworks that integrate neutrosophic and plithogenic extensions. The study by Mukherjee et al. [63] introduces a generalized interval-valued neutrosophic rough soft set framework that effectively handles uncertainty and demonstrates its practical use in water quality assessment. Collectively, these contributions consolidate the theoretical backbone and applied versatility of FSS theory and its extensions, making them indispensable for analyzing complex networks, digital security assessments, cyber risk evaluation, and social systems under uncertainty.

Table-1: Literature Review on Neutrosophic Set

Authors	Year	Environment	Model Type	Area of Application
Broumi et al. [9]	2018	Neutrosophic Set	Overview of Neutrosophic Theory	Theoretical Foundations
Das et al. [26]	2022	Single-Valued Pentapartitioned Neutrosophic Set	Lie Algebra in Neutrosophic Environment	Algebraic Structures
Das et al. [27]	2023	Neutrosophic Set	Neutrosophic d-Filter in d-Algebra	Algebra, Mathematical Logic
Das et al. [64]	2024	Water Quality Assessment	Neutrosophic Approach to Water Quality	Environmental Science, Water Quality Management
Deli et al. [10]	2015	Neutrosophic Multisets	Application in Medical Diagnosis	Medical Diagnosis
Dey et al. [16]	2016	Bipolar Neutrosophic	TOPSIS for Multi-Attribute Decision Making	Decision Making under Neutrosophic Environment
Otay & Kahraman [29]	2019	Neutrosophic Set Theory	Review of Neutrosophic MCDM Models	Multi-Criteria Decision-Making
Peng & Dai [30]	2020	Neutrosophic Set Theory	Bibliometric Analysis of Neutrosophic Sets	Literature Review, Artificial Intelligence
Pramanik et al. [17]	2018	Bipolar Neutrosophic Set Environment	TODIM Method for Group Decision-Making	Group Decision-Making
Pramanik et al. [15]	2018	Bipolar Neutrosophic Set Environment	VIKOR-Based MAGDM Strategy	Multi-Attribute Group Decision-Making

Pramanik [31]	2020	Rough Neutrosophic Set	Overview of Rough Neutrosophic Theory	Theoretical Foundations, Uncertainty Analysis
Pramanik [32]	2022	Single-Valued Neutrosophic Set	Overview of Single-Valued Neutrosophic Theory	Theoretical Foundations, Decision Science
Pramanik et al. [65]	2023	SVNS Environment	Neutrosophic BWM-TOPSIS Strategy	Multi-Attribute Decision-Making
Pramanik et al. [28]	2018	Neutrosophic Environment	Contributions of Indian Researchers to MADM	Bibliometric Review, Decision-Making
Shil et al. [18]	2022	SVNS Environment	Hyperbolic Cosine Similarity Measure-Based MADM	Multi-Attribute Decision-Making
Smarandache [1]	1998	Neutrosophic Logic	Foundations of Neutrosophic Logic & Probability	Theoretical Foundations
Smarandache [2]	2005	Neutrosophic Set	Generalization of Intuitionistic Fuzzy Sets	Theoretical Foundations
Smarandache [3]	2007	Neutrosophic Logic	Unifying Field in Logic	Theoretical Foundations
Smarandache & Pramanik [65]	2016	Neutrosophic Set	New Trends in Neutrosophic Theory	Theoretical Developments
Smarandache & Pramanik [23]	2018	Neutrosophic Set	New Trends in Neutrosophic Theory (Vol.2)	Theoretical Developments
Smarandache [4]	2019	Neutrosophic Set	Generalization of Various Fuzzy Sets	Theoretical Foundations
Smarandache [6]	2013	Refined Neutrosophic Logic	Applications to Physics	Applied Neutrosophic Logic
Smarandache [7]	2020	Neutrosophic Set	Score, Accuracy, and Certainty Functions	Decision-Making Models
Smarandache [21]	2016	Neutrosophic SuperHyper Algebra	SuperHyper Algebra & Neutrosophic SuperHyper Algebra	Algebra, Mathematical Structures
Smarandache [67]	2018	Neutrosophic Sets & Systems	Plithogenic Hypersoft Set	Soft Set Theory, Decision-Making
Smarandache [25]	2022	Neutrosophic Sets & Systems	SuperHyperFunction & Neutrosophic SuperHyperFunction	Functional Analysis, Mathematical Modeling
Smarandache [11]	2023	Neutrosophic Systems with Applications	SuperHyperSoft Set & Fuzzy Extension	Fuzzy Systems, Soft Set Theory
Smarandache [12]	2023	Neutrosophic Sets & Systems	Multi Neutrosophic Set	Theoretical Foundations
Ye & Ye [19]	2014	Single-Valued Neutrosophic Multiset	Dice Similarity Measures	Medical Diagnosis

Table-2: Literature Review on Neutrosophic Graph Theory

Authors	Year	Environment	Model Type	Area of Application
Akram & Shahzadi [24]	2017	Single-Valued Neutrosophic Set	Neutrosophic Graphs	Graph Theory, Uncertainty Modeling
Broumi et al. [8]	2016	Single-Valued Neutrosophic Set	Neutrosophic Graphs	Graph Theory
Garrett [34]	2022	Neutrosophic Graphs	Extensions of Neutrosophic Graphs	Graph Theory
Garrett [20]	2022	Neutrosophic Hypergraphs	Co-degree and Degree Analysis of Neutrosophic Hypergraphs	Graph Theory
Shah & Hussain [33]	2016	Neutrosophic Soft Set	Neutrosophic Soft Graphs	Graph Theory
Shannon & Atanassov [35]	1994	Intuitionistic Fuzzy Set	Intuitionistic Fuzzy Graphs	Graph Theory
Smarandache [36]	2020	Neutrosophic Sets & Systems	n-SuperHyperGraph & Plithogenic n-SuperHyperGraph	Graph Theory, Algebraic Structures
Smarandache [22]	2024	Neutrosophic Sets & Systems	SuperHyperStructure & Neutrosophic SuperHyperStructure	Algebraic Structures, Graph Theory

This chapter discusses the topological properties of NHEWSHGs and gives special attention to connectivity, cycle formation, and clustering. We would find potential patterns, critical pathways, and cohesive structures that collectively define complex networks in the form of these properties under a neutrosophic framework. We examine the effect of neutrosophic weights on network topology and functionality through theoretical propositions, numerical examples, and algorithmic analyses. The value of this research is that it can help reveal network behavior in uncertain environments. With our formal framework based on modeling uncertainty and partial truth, we can apply this to a wide range of applications, from social networks, biological systems, and collaborative platforms. Understanding complex systems made by studying NHEWSHGs ultimately helps with a deeper insight into complex systems and leads to more robust networks in the future. In this work, the structure of the whole work is done as follows: Section 2 is devoted to the deep analysis of some basic concepts like Neutrosophic sets, superhyperstructures, Neutrosophic SuperHyperStructures, SuperHyperGraphs, and their properties. Section 3 gives the topological properties and possible applications of NHEWSHG in social Networks. In the last section, Section 5 concludes our study by summarizing our key findings and suggesting directions for future research.

2. Preliminaries

This segment revisits fundamental concepts we can't help but recall for the next section. The definitions and properties include neutrosophic sets, superhyperstructures, Neutrosophic SuperHyperStructures, SuperHyperGraphs, and their related operators. Let S be a non-empty set.

Definition 2.1 ([1], [2])**.** For any subset X of S, A neutrosophic set (NSS) X on S is defined as follows:

$$X = \{(\alpha, T_X(\alpha), I_X(\alpha), F_X(\alpha)) \mid \alpha \in S\}$$

where $T_X, I_X, F_X: S \to [0, 1]$ represent the truth, indeterminacy, and falsity membership functions, respectively. These functions must satisfy:

$$0 \leq T_X(\alpha) + I_X(\alpha) + F_X(\alpha) \leq 3, \qquad \text{for all } \alpha \in S.$$

Definition 2.2 ([22]). A SuperHyperStructure is defined on the n-th powerset $P^n(S)$ of S, where n is an integer greater than or equal to 1. This structure features SuperHyperOperators that are defined as follows:

$$\#SHS: (P^r(S))^m \to P^n(S),$$

were $P^r(S)$ represents the r-powerset of S, where r is an integer greater than or equal to 1. Similarly, $P^n(S)$ denotes the n-th powerset of S.

Definition 2.3 ([22]). A Neutrosophic SuperHyperStructure is a structure defined on the n-th powerset $P^n(S)$ of S, where n is an integer greater than or equal to 1. It features Neutrosophic SuperHyperOperators denoted as $\#SHS: (P^r(S))^m \to P^n(S)$, where $P^r(S)$ represents the r-th powerset of S for integer $r \geq 1$, and $P^n(S)$ denotes the n-th powerset of S.

These operators map from m instances of r-powersets to the n-th powerset of S.

Definition 2.4 ([36])**.** An ordered pair $(G \subseteq P(V), E \subseteq P(V))$ is called a SuperHyperGraph and is denoted by SHG:

Definition 2.5 ([36]). An ordered pair $(G_n \subseteq P_n(V), E_n \subseteq P_n(V))$ is called an n-SuperHyperGraph and is denoted by n-SHG:

3. Topological Properties of NHEWSHGs and their applications in Social Networks

In this section, we delve into the topological properties of NHEWSHGs and explore their practical applications in social networks.

Definition 3.1. A NHEWSHG is a mathematical structure $\Omega = (G, E, W)$ where:

- $G \subseteq P^n(V)$ is the set of subsets of nodes involved in hyperedges, where $P^n(V)$ is the n-th power set of the set of vertices V.
- $E \subseteq P^n(V)$ is the set of hyperedges, where each hyperedge is a subset of $P^n(V)$.
- W is a function that assigns a neutrosophic weight to each hyperedge, represented as $W(e) = (T_e, I_e, F_e)$, where T_e, I_e, and F_e are the truth, indeterminacy, and falsity components, respectively, satisfying $0 \leq T_e + I_e + F_e \leq 3$.

Example 3.2. Consider a NHEWSHG representing a social network where nodes correspond to individuals and hyperedges represent groups or interactions among individuals. We'll use neutrosophic weights to capture uncertainty and partial truth in the relationships. Let $V = \{A, B, C\}$ represent the set of individuals in the social network.

Nodes: Let $G = \{\{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}\} \subset P(V)$.

Hyperedges: The hyperedges E are subsets of $P(V)$, the powerset of V, denoting groups of individuals.

- $H1 = \{\{A, B\}, \{B, C\}\}$: Represents a group consisting of either A and B or B and C.
- $H2 = \{\{B, C\}, \{C\}\}$: Represents a group consisting of either B and C or just C.

Neutrosophic Weights: The neutrosophic weights $W(e) = (T_e, I_e, F_e)$ for each hyperedge e are as follows:

- $W(H1) = (0.7, 0.2, 0.1)$: Indicates a strong positive influence (0.7) with some uncertainty (0.2) and minor falsity (0.1) in the group represented by $H1$.
- $W(H2) = (0.6, 0.3, 0.1)$: Indicates a moderately positive influence (0.6) with higher uncertainty (0.3) and minor falsity (0.1) in the group represented by $H2$.

Using the second powerset of the set of individuals, the NHEWSHG captures the intricate linkages in the social network in this example, enabling a more complex depiction of uncertain and partially true connections among individuals.

Definition 3.3. A NHEWSHG $\Omega = (G, E, W)$ is said to be *connected* if there exists a sequence of hyperedges such that any two vertices u and v in the graph can be linked through a path of hyperedges, taking into account the neutrosophic weights of these hyperedges.

Considering the uncertainties and indeterminacies reflected by neutrosophic weights, connectivity in a NHEWSHG is the extent to which the vertices (nodes) are connected through hyperedges. To characterize it correctly, we must expand the traditional connection definition to include the neutrosophic framework.

Path and Neutrosophic Weights: A *path* between two vertices u and v is a sequence of

hyperedges $e_1, e_2, \ldots, e_k$ such that $u \in e_1$, $v \in e_k$, and each consecutive pair of hyperedges e_i and e_{i+1} share at least one common vertex.

Formally, a path P between u and v can be represented as:

$$P = (e_1, e_2, \ldots, e_k) \quad \text{where} \quad u \in e_1 \text{ and } v \in e_k \quad \text{and} \quad e_i \cap e_{i+1} \neq \emptyset \quad \text{for} \quad 1 \leq i < k.$$

The *neutrosophic weight* of a path P is the combined neutrosophic weight of the hyperedges in the path, calculated through appropriate neutrosophic operations. If $W(e_i) = (T_i, I_i, F_i)$, the neutrosophic weight of the path can be defined as:

$$W(P) = (T_P, I_P, F_P), \quad \text{where} \quad T_P = \prod_{i=1}^{k} T_i, \; I_P = \prod_{i=1}^{k} I_i, F_P = \prod_{i=1}^{k} F_i.$$

Connectivity Measure: We define an *overall neutrosophic connectivity* measure $C(\Omega)$ that integrates the truth, indeterminacy, and falsity components of neutrosophic weights to provide a single value representing the connectivity of the NHEWSHG.

Formally, the overall neutrosophic connectivity measure $C(\Omega)$ can be defined as:

$$C(\Omega) = \frac{1}{\big(|G|(|G|-1)\big)} \sum_{(u,v) \in GXG, u \neq v} T_{P_{uv}},$$

where P_{uv} is the optimal path between u and v.

In NHEWSHGs, neutrophilic weights are essential for defining connection. By influencing path selection, aggregating weights along paths, determining path strength, taking threshold considerations into account, and enabling a nuanced assessment of connectivity that takes uncertainties and partial truths in the relationships between nodes and hyperedges into account, neutrophilic weights in neutrophilic hyperedge-weighted super hypergraphs have an impact on connectivity.

Example 3.4. Consider a NHEWSHG representing a social network where nodes correspond to individuals and hyperedges represent connections or interactions among individuals. Let $V = \{A, B, C, D\}$ represent the set of individuals in the social network.

Nodes: Let $G = \{\{A\}, \{B\}, \{C\}, \{D\}, \{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\} \subset P(V)$.

Hyperedges: The hyperedges E are subsets of $P(V)$, the powerset of V, denoting groups of individuals.

- $H1 = \{\{A, B\}, \{B, C\}\}$: Represents a group consisting of either A and B or B and C.
- $H2 = \{\{B, C\}, \{C, D\}\}$: Represents a group consisting of either B and C or C and D.

Neutrosophic Weights: The neutrosophic weights $W(e) = (T_e, I_e, F_e)$ for each hyperedge e are as follows:

- $W(H1) = (0.7, 0.2, 0.1)$: Indicates a strong positive influence (0.7) with some uncertainty (0.2) and minor falsity (0.1) in the group represented by $H1$.
- $W(H2) = (0.6, 0.3, 0.1)$: Indicates a moderately positive influence (0.6) with higher uncertainty (0.3) and minor falsity (0.1) in the group represented by $H2$.

Connectivity Analysis: To determine if the NHEWSHG is connected, we analyze the paths between vertices using neutrosophic weights:

- There exists a path between nodes {A} and {B} through the hyperedge $H1$.
- There exists a path between nodes {A} and {C} through the hyperedges $H1$.
- There exists a path between nodes {A} and {D} through the hyperedges $H1$ and $H2$.
- There exists a path between nodes {B} and {C} through the hyperedge $H1$.
- There exists a path between nodes {B} and {D} through the hyperedges $H2$.
- There exists a path between nodes {C} and {D} through the hyperedge $H2$.
- There exists a path between nodes {A, B} and {A, C} through the hyperedges $H1$.
- There exists a path between nodes {A, B} and {A, D} through the hyperedges $H1$ and $H2$.
- There exists a path between nodes {A, B} and {B, C} through the hyperedge $H1$.
- There exists a path between nodes {B, C} and {C, D} through the hyperedge $H2$.

The neutrosophic weights of these paths are:

$$\begin{aligned} W(\{A\},\{B\}) &= (0.7, 0.2, 0.1) \\ W(\{A\},\{C\}) &= (0.7, 0.2, 0.1) \\ W(\{A\},\{D\}) &= (0.42, 0.06, 0.01) \\ W(\{B\},\{C\}) &= (0.7, 0.2, 0.1) \\ W(\{B\},\{D\}) &= (0.6, 0.3, 0.1) \\ W(\{C\},\{D\}) &= (0.6, 0.3, 0.1) \\ W(\{A,B\},\{A,C\}) &= (0.49, 0.04, 0.01) \end{aligned}$$

$$W(\{A,B\},\{A,D\}) = (0.42, 0.06, 0.01)$$
$$W(\{A,B\},\{B,C\}) = (0.7, 0.2, 0.1)$$
$$W(\{B,C\},\{C,D\}) = (0.6, 0.3, 0.1)$$

Total Neutrosophic Connectivity Measure: Now, compute $C(\Omega)$ using these calculated weights.

$$C(\Omega) = \frac{1}{(|G|(|G|-1))} \sum_{(u,v)\in GXG, u\neq v} T_{P_{uv}}$$

Here, $|G| = 10$, hence $|G|(|G|-1) = 10 \times 9 = 90$.

Sum of truth values T: $0.7 + 0.7 + 0.42 + 0.7 + 0.6 + 0.6 + 0.49 + 0.42 + 0.7 + 0.6 = 5.93$

Calculate the overall measure:

$$C(\Omega) = \frac{5.93}{90} = 0.0659$$

Thus, the neutrosophic connectivity measure $C(\Omega)$ for the NHEWSHG with the updated vertex set is approximately 0.0659.

Definition 3.5. A cycle in a NHEWSHG $\Omega = (G, E, W)$ is a sequence of hyperedges $e_1, e_2, \ldots, e_k$ such that:

- The first vertex of e_1 is the same as the last vertex of e_k:

$$u \in e_1 \quad \text{and} \quad u \in e_k$$

- Each consecutive pair of hyperedges e_i and e_{i+1} in the cycle shares at least one common vertex:

$$e_i \cap e_{i+1} \neq \emptyset \quad \text{for } i = 1, 2, \ldots, k-1$$

The neutrosophic weight of the cycle $W(e_1, e_2, \ldots, e_k)$ is calculated based on the neutrosophic weights of the individual hyperedges in the cycle. Let $W(e_i) = (T_i, I_i, F_i)$ for $i = 1, 2, \ldots, k$, then:

$$W(e_1, e_2, \ldots, e_k) = \left(\prod_{i=1}^{k} T_i, \prod_{i=1}^{k} I_i, \prod_{i=1}^{k} F_i\right)$$

In NHEWSHGs, cycles represent closed paths or loops formed by hyperedges that connect vertices within the graph. A cycle is a sequence of hyperedges $e_1, e_2, \ldots, e_k$ where the first vertex of e_1 is the same as the last vertex of e_k.

Neutrosophic Cycle Weight: The neutrosophic weight of a cycle is determined by aggregating the neutrosophic weights of its constituent hyperedges. Given hyperedges $e_1, e_2, \ldots, e_k$

with weights $W(e_i) = (T_i, I_i, F_i)$, the neutrosophic weight of the cycle is:

$$W(e_1, e_2, \ldots, e_k) = \left(\prod_{i=1}^{k} T_i, \prod_{i=1}^{k} I_i, \prod_{i=1}^{k} F_i\right)$$

In NHEWSHGs, cycles are crucial for comprehending intricate feedback loops and linkages. They aid in analyzing network dynamics, identifying patterns, and assessing the graph's general dependability and connectivity in the face of uncertainty.

Cycle generation in NHEWSHGs is strongly influenced by indeterminacy and partial truth, which provide flexibility in cycle designs, allow for uncertainty in hyperedge arrangements, and increase the number of possible alternate paths.

Cycle existence and strength are impacted by partial truth; higher truth components indicate stronger links. The total effect, uncertainty, and complexity of cycles are thus reflected in neutrosophic cycle weights, which are influenced by partial truth and indeterminacy. Accurately examining connection patterns and linkages within these graphs requires understanding these impacts.

Example 3.6. Consider a NHEWSHG representing a social network. Let $V = \{A, B, C, D\}$ represent the set of individuals in the social network.

Nodes: Let $G = \{\{A\},\{B\},\{C\},\{D\},\{A,B\},\{A,C\},\{A,D\},\{B,C\},\{B,D\},\{C,D\}\} \subset P(V)$.

Hyperedges: The hyperedges E are subsets of $P(V)$, the powerset of V, denoting groups of individuals. The neutrosophic weights for each hyperedge are as follows:

- $H1 = \{\{A, B\}, \{B, C\}\}$: Neutrosophic weight $W(H1) = (0.7, 0.2, 0.1)$
- $H2 = \{\{B, C\}, \{C, D\}\}$: Neutrosophic weight $W(H2) = (0.6, 0.3, 0.1)$
- $H3 = \{\{D, A\}, \{A, C\}\}$: Neutrosophic weight $W(H3) = (0.8, 0.1, 0.1)$.

Now, let's identify a cycle in this graph:

Cycle: Consider the cycle $C = \{\{A,B\},\{B,C\},\{C,D\},\{D,A\}\}$.

The neutrosophic weight of cycle C is calculated as follows:

$$W(C) = W(H1) \odot W(H2) \odot W(H3) = (0.7 \cdot 0.6 \cdot 0.8, 0.2 \cdot 0.3 \cdot 0.1, 0.1 \cdot 0.1 \cdot 0.1) = (0.336, 0.006, 0.001)$$

This cycle C has a neutrosophic weight $W(C) = (0.336, 0.006, 0.001)$, indicating its overall influence, uncertainty, and falsity within the graph.

Definition 3.7. Clustering in NHEWSHGs refers to identifying and analyzing groups or clusters of vertices that exhibit strong connections or interactions. Formally, a cluster $C \subseteq V$ is a subset of vertices such that the following properties hold:

Characteristics of Clusters:

Internal Cohesion: For any pair of vertices $u, v \in C$, there exists a hyperedge $e \in E$ such that $u, v \in e$ and the neutrosophic weight $W(e) = (T_e, I_e, F_e)$ has a high truth component T_e indicating strong influence and high internal connectivity within the cluster.

External Separation: For any vertex $u \in C$ and any vertex $v \notin C$, if there exists a hyperedge $e \in E$ such that $u, v \in e$, the neutrosophic weight $W(e) = (T_e, I_e, F_e)$ has a lower truth component T_e and higher indeterminacy component I_e, indicating weaker connections between clusters.

Neutrosophic Weight Analysis: Let $W(C)$ denote the set of neutrosophic weights of hyperedges within the cluster C. Clustering involves identifying clusters such that the average truth component T_C of $W(C)$ is maximized, representing cohesive and influential relationships within the cluster.

Formally, for a given cluster C, $\overline{T_C} = \frac{1}{|E(C)|}\sum_{e \in E(C)} T_e$,

where $E(C) = \{e \in E \mid \forall u, v \in C, u, v \in e\}$.

Clustering in NHEWSHGs improves comprehension of intricate interactions under ambiguity and partial truth by revealing information about the organization, connectivity patterns, and cohesive substructures inside the graph.

Example 3.8. Consider a social network represented by a NHEWSHG. Let the set of people in the social network be represented by $V = \{A, B, C, D\}$.

Nodes: Let $G = \{\{A\}, \{B\}, \{C\}, \{D\}, \{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\} \subset P(V)$.

Hyperedges: Groups of persons are represented by the hyperedges E, which are subsets of P(V), the powerset of V. Let's determine the neutrosophic weights of the following hyperedges:

- $H1 = \{\{A, B\}, \{B, C\}\}$: Neutrosophic weight $W(H1) = (0.7, 0.2, 0.1)$
- $H2 = \{\{B, C\}, \{C, D\}\}$: Neutrosophic weight $W(H2) = (0.6, 0.3, 0.1)$
- $H3 = \{\{D, A\}, \{A, C\}\}$: Neutrosophic weight $W(H3) = (0.8, 0.1, 0.1)$

Now, let's analyze clustering within this graph:

Clustering Analysis:

- **Cluster 1**: Let $\{A, B, C\}$ be the set of vertices bearing the hyperedges H1 and H2. Given that both hyperedges have high truth components (T) and low indeterminacy (I) and falsity (F) components, the neutrosophic weights inside this cluster suggest strong internal cohesiveness.

$$W(H1) = (0.7, 0.2, 0.1), \qquad W(H2) = (0.6, 0.3, 0.1)$$

- **Cluster 2**: Another cluster is formed by the vertices {C, D} and hyperedge H2. Cohesion within the cluster is still visible even if the truth component in H2 for this cluster is comparatively weaker.

$$W(H2) = (0.6, 0.3, 0.1)$$

External Connections: Vertex C is the link between clusters 1 and 2, suggesting a weaker exterior connection than the robust internal connections inside each cluster.

Neutrosophic Weight Analysis:

- ❖ Information on the cohesiveness and impact of clusters can be gleaned from their neutrosophic weights. Higher truth components and lower indeterminacy and falsity components indicate stronger and more dependable relationships.
- ❖ Clustering analysis in NHEWSHGs is aided by identifying clusters and their internal structures through the analysis of neutrosophic weights of hyperedges.
- ❖ Using the strengths and uncertainties of links within the graph, clustering analysis in NHEWSHGs finds cohesive substructures, as seen in this example.

4. Impact of Neutrosophic Weights on Clustering Coefficients and Community Structures

4.1 Clustering Coefficients

A key metric in network analysis is the clustering coefficient, which expresses how much a graph's nodes prefer to cluster together. The neutrosophic weights assigned to edges considerably impact the clustering coefficient in the context of neutrosophic networks, where partial truth values and uncertainty influence connectivity.

Strong clustering and high truth components: The clustering coefficient rises with a high neutrosophic weight truth component T. This suggests a greater likelihood of the nodes forming close-knit communities, which would strengthen local connectedness. Stronger, more distinct clusters result from relationships with a larger truth value in real-world applications like social networks or biological systems.

Higher truth component T → Higher clustering coefficient

Effect of Indeterminacy and Falsity: Conversely, the clustering coefficient is weakened by larger indeterminacy (I) or falsity (F) levels. Clustering becomes less clear when indeterminacy is introduced because it creates doubt regarding the existence or strength of linkages. Strong local clustering is also less likely when the high falsehood score indicates deceptive or poorly supported relationships. For instance, unclear or inaccurate user interactions reduce the dependability of proposed clusters in recommendation algorithms.

Higher indeterminacy I or falsity F → Lower clustering coefficient.

4.2 Community Structures

The network's community structure reflects the distribution of nodes into groups with comparatively few intergroup connections and extensive intragroup connections. Neutrophilic weights influence the formation and preservation of the structural integrity of communities.

Strong Intra-Community Connections: Well-defined clusters result from stronger intra-community links when the truth component T is high. A community's relationships stay strong and meaningful when its truth value is high, which promotes strong cohesion among its members. This is seen in real-world settings when there are close relationships among professionals in professional networks.

High truth component T → Strong intra-community connections

Inter-Community Connections and the Role of Truth Components: Stronger linkages between communities result from high truth components, improving information flow throughout the network. For instance, robust intercommunity ties facilitate successful interdisciplinary research collaborations in knowledge-sharing systems.

Higher truth components between communities → Stronger inter-community connections

Disruptive Influence of Indeterminacy and Falsity: Community systems can be upset by high degrees of indeterminacy (I) or falsehood (F), which introduce uncertainty or improper linkages. The network becomes fragmented due to the weakened intra- and inter-community ties. In social dynamics, conflicting interests or false information can disrupt a group's cohesiveness.

Indeterminacy or falsity → Disrupts community structures, weaker connections

4.3 Neutrosophic Weight Analysis and Its Influence

Analyzing community structures and clustering coefficients requires an understanding of neutrosophic weights. The interaction of truth, indeterminacy, and falsehood determines the strength and integrity of network architectures.

Influence of Higher Truth Components: Stronger and more cohesive clusters are directly correlated with a greater truth component (T), which strengthens the network's local and global connections. Communities are kept resilient and well-formed as a result.

Higher truth component T → Stronger and more cohesive clusters

Effect of Indeterminacy and Falsity: The overall network structure deteriorates when falsehood (F) or indeterminacy (I) rises. Communities become less stable as clustering coefficients decline, increasing the system's vulnerability to inefficiency or fragmentation.

Indeterminacy or falsity → Weakened connections affect clustering and communities.

Proposition 1 (Connectivity): Consider the NHEWSHG $\Omega = (G, E, W)$. The connectivity between two vertices u and v in Ω is determined by the aggregate neutrosophic weight of all possible paths between u and v.

Proof: We consider the total neutrosophic weight of all potential routes between vertices u and v in Ω to demonstrate Proposition 1. Let P (u, v) represent the collection of all potential routes from u to v. The link between u and v's aggregate neutrosophic weight is as follows:

$$W(u, v) = \sum_{p \in P(u,v)} W(p),$$

where W (p) means path p's neutrosophic weight.

Now, let's examine the components of $W(u, v)$:

- $T(u, v)$: Represents the truth component of the connectivity between u and v. The overall strength and dependability of connections along all potential paths are captured.
- $I(u, v)$: Denotes the connectivity's indeterminacy component. The robustness of the connection is impacted by indeterminacy, which represents uncertainties or ambiguities in the routes.
- $F(u, v)$: Stands for the connectivity's falsehood component. It takes into consideration possible erroneous connections or path faults.

A thorough assessment of the connectivity between u and v is given by the aggregate neutrosophic weight $W(u, v)$, which takes into account both the uncertainties/potential falsehood (indeterminacy/falsity) and the strength of connections (truth). Consequently,

Proposition 1 is valid since the total neutrosophic weight of all potential paths between two vertices u and v in a NHEWSHG Ω does, in fact, determine the connectedness between them.

Theorem 1 (Cycle Existence): The NHEWSHG Ω = (G, E, W) has a cycle if and only if there is a sequence of hyperedges $e_1, e_2, \ldots, e_k$ such that:

- The first vertex of e_1 is the same as the last vertex of e_k.
- At least one vertex is shared by every successive pair of hyperedges in the cycle, e_i and e_{i+1}.
- The cycle $W(e_1, e_2, \ldots, e_k)$ has an aggregate neutrosophic weight that is not zero.

Proof. We assume that the conditions required for the existence of a cycle in a NHEWSHG $\Omega = (G, E, W)$.
First, we have a cycle in Ω if there is a succession of hyperedges $e_1, e_2, \ldots, e_k$ that meet the specified criteria. In the graph, this series of hyperedges creates a closed loop where each successive pair shares at least one vertex, and the initial vertex of e1 is the same as the last vertex of ek. Furthermore, the cycle has an influence or presence inside the graph because its aggregate neutrosophic weight is non-zero.
On the other hand, a cycle in Ω indicates a series of hyperedges that meet the requirements of the theorem. Based on connection and aggregate neutrosophic weight, this sequence creates a closed loop with the desired characteristics, proving the existence of the cycle.
The requirements for cycle existence in NHEWSHGs are thus established by Theorem 1, which connects the aggregate neutrosophic weights of cycles to the graph's topological characteristics.

Proposition 2 (Clustering Coefficients): In a NHEWSHG, the neutrosophic weights of the hyperedges attached to a vertex affect its clustering coefficient. Higher clustering coefficients for the vertex are a result of higher truth components in these neutrosophic weights.

Theorem 2 (Community Structure): Communities arise as groups of vertices with weaker external connections (lower truth components or higher indeterminacy/falsity components in inter-community hyperedges) and strong internal connections (high truth components in hyperedge weights) in a NHEWSHG Ω = (G, E, W).

Proof. We examine the characteristics of hyperedge weights $W(e) = (T_e, I_e, F_e)$ to demonstrate the formation of communities in neutrophilic hyperedge-weighted super hypergraphs.
As shown by high truth components (T_e) in the hyperedge weights of intra-community hyperedges, vertices that share strong internal connections define a community inside the graph. These high truth components indicate strong and dependable relationships between the community's vertices.

Conversely, hyperedges with larger indeterminacy/falsity components (I_e or F_e) or lower truth components represent community exterior linkages. In contrast to the strong internal connections inside communities, these inter-community hyperedges may show uncertainty or be erroneous connections, resulting in weaker outward linkages.

Thus, in NHEWSHGs, the distribution of hyperedge weights controls the establishment of communities, where weaker outward connections and communities that are cohesive by strong internal connections.

As a result, communities in NHEWSHGs appear as vertex clusters with smaller truth components or larger indeterminacy/falsity components in external hyperedges and higher truth components in interior hyperedges.

Theorem 3 (Neutrosophic Cycle Weight): A cycle's neutrosophic weight in a NHEWSHG is

calculated by adding the neutrosophic weights of its individual hyperedges, which represent the cycle's total effect, degree of uncertainty, and possible falsity.

Proof. To demonstrate the claim, consider a cycle C in a NHEWSHG Ω = (G, E, W), where W gives hyperedges neutrosophic weights.

The sum of the neutrosophic weights of cycle C's component hyperedges can be used to calculate the neutrosophic weight W (C):

$$W\ (C) = \sum_{e \epsilon C} W\ (e)$$

where $W(e)$ represents the neutrosophic weight of hyperedge e in the cycle C.

This aggregation procedure captures the entire impact of the cycle inside the graph. The strength and dependability of the links within the cycle are reflected in the truth component T (C) of W (C). The degree of uncertainty and possible erroneous connections related to the cycle are indicated by the indeterminacy I(C) and falsity F(C) components of W (C), respectively.

W (C) thoroughly assesses the cycle's impact by summing the neutrosophic weights of its component hyperedges, considering the cycle's influence, uncertainty, and possible falsehood within the NHEWSHG G.

Consequently, a cycle's neutrosophic weight in a NHEWSHG is calculated by adding up the neutrosophic weights of its individual hyperedges, which representing the cycle's total influence, uncertainty, and potential falsity.

4.4 Social Network Example

Consider a social network with five individuals: Alice (A), Bob (B), Carol (C), Dave (D), and Eve (E). A NHEWSHG G = (V, E, W) represents their relationships, where:

- $V = \{A, B, C, D, E\}$ represents the vertices (individuals).
- E contains hyperedges representing relationships.
- $W\ (e) = (T_e, I_e, F_e)$ assigns neutrosophic weights to hyperedges, capturing uncertainty and indeterminacy.

1. **Relationships**:

$$E = \{\{A, B\}, \{B, C\}, \{C, D\}, \{D, E\}, \{E, A\}\}$$

Each hyperedge represents a relationship between two individuals.

2. **Neutrosophic Weights**:

$$W(\{A, B\}) = (0.7, 0.2, 0.1)$$

$$W(\{B, C\}) = (0.9, 0.1, 0)$$

$$W(\{C, D\}) = (0.6, 0.3, 0.1)$$

$$W(\{D, E\}) = (0.8, 0.2, 0)$$

$$W(\{E, A\}) = (0.5, 0.4, 0.1)$$

3. **Connectivity Analysis**:

- Alice (A) is connected to Bob (B), who is connected to Carol (C), forming a chain.
- Dave (D) is connected to Eve (E), and there's also a connection between Eve (E)

and Alice (A), completing a cycle.

4. **Cycle Formation**:
   - $Cycle\ 1$: $\{A,B\}\ \{B,C\}\ \{C,D\}\ \{D,E\}\ \{E,A\}$
   - $Cycle\ 2$: $\{A,B\}\ \{E,A\}\ \{D,E\}\ \{C,D\}\ \{B,C\}$
5. **Neutrosophic Weight of Cycles**:

$$W\ (Cycle\ 1)\ =\ (3.5, 1.2, 0.3)$$

$$W\ (Cycle\ 2)\ =\ (3.5, 1.2, 0.2)$$

Clustering:

1. **Community Formation**:
   - Community 1: Alice (A), Bob (B), and Carol (C) form a cluster with strong internal connections.
   - Community 2: Dave (D) and Eve (E) form another cluster.
2. **External Connections**: Although Eve (E) and Alice (A) have a connection that unites the two communities, it has a lower truth component, suggesting fewer external ties.

This example shows how NHEWSHGs can be used to study connectedness, cycles, and clustering in a social network while considering uncertain and indeterminate links.

4.5 Applications in Research Collaboration Network

We now present a novel approach for building and analyzing hyperedge-weighted superhypergraphs using neutrosophic logic. By integrating degrees of truth, falsehood, and indeterminacy into group dynamics, this algorithm allows for sophisticated analysis and visualization while capturing the subtleties and complexity of social network relationships.

Algorithm 1: NHEWSHGs for Data Representation

Step 1: Preparing the Data

Gather information from a social network where different levels of indeterminacy, falsehood, and truth mark ties between people. Each relationship should be represented as a neutrosophic value made up of indeterminacy (I), falsehood (F), and truth (T).

Step 2: Building Neutrosophic HyperEdges

Define hyperedges to symbolize relationships inside a group. For instance, a hyperedge might stand for a cooperative team or a group of friends. Based on the connections inside the group, give hyperedges neutrosophic weights. To determine the hyperedge weight, add the neutrosophic values of each unique relationship.

Step 3: Creating the SuperHyperGraph

Create a SuperHyperGraph by assembling the Neutrosophic HyperEdges, in which nodes stand for people and hyperedges for group ties. Add neutrosophic weights to hyperedges to account for the different levels of truth, falsehood, and indeterminacy in group dynamics.

Step 4: Visualization and Analysis

Analyze connectedness, cycles, and clustering using algorithms created for NHEWSHGs. Show the graph structure visually, emphasizing areas of uncertainty, possible contradictions, and clusters with

dominant truth values.

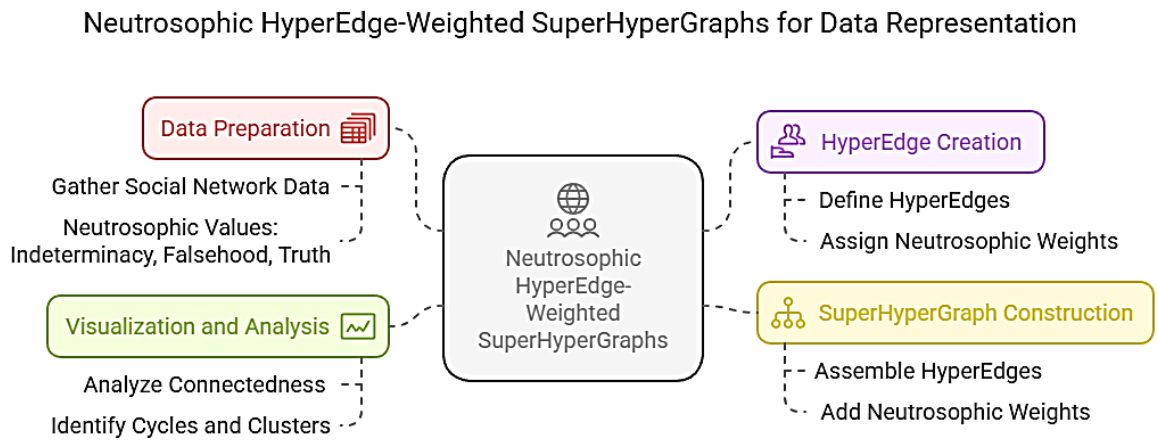

Fig. 1: NHEWSHGs for Data Representation

Example 4.6: Assume that a university has a network of research collaborations that spans multiple departments and researchers. We gather information about researcher collaborations, such as the projects they collaborated on and the quality of such collaborations. Let's examine Alice, Bob, and Carol, three researchers, to keep things simple.

Bob and Alice worked together on Project A and Project B. With a neutrosophic rating of (0.9T, 0.05F, 0.05I), which indicates excellent trust and dependability, they worked well together on Project A. The neutro-sophic value of their cooperation on Project B was 0.7T, 0.2F, and 0.1I, indicating a moderate level of trust but some uncertainty.

With a neutrosophic value of (0.6T, 0.3F, 0.1I), Bob and Carol worked together on Project A, which showed moderate trust but some falsehood and indeterminacy.

To symbolize cooperative teams or projects, we define hyperedges. We have two hyperedges in this situation:

Project A, which involves Alice, Bob, and Carol, is represented by Hyperedge 1. The neutrosophic values of the individual collaborations— (0.9T, 0.05F, 0.05I) for Alice-Bob, and (0.6T, 0.3F, 0.1I) for Bob-Carol—are used to calculate the neutrosophic weight for this hyperedge.

Hyperedge 2, including Alice and Bob, stands in for Project B. According to their collaboration, the neutrosophic weight for this hyperedge is (0.7T, 0.2F, 0.1I).

We create a SuperHyperGraph by assembling the Neutrosophic HyperEdges:

- ❖ Nodes represent individual researchers Alice, Bob, and Carol.
- ❖ Project A and Project B are examples of collaborative initiatives represented by hyperedges.
- ❖ The different levels of trust, falsehood, and indeterminacy within each project cooperation are captured by neutrophilic weights on hyperedges.

We analyze the network using algorithms that are designed for NHEWSHGs.

- ❖ Connectivity Analysis: In addition, we analyze connectivity between researchers across projects.
- ❖ Clustering Analysis: It reports the cohesive groups of researchers by examining clustering coefficients and community structures
- ❖ Collaboration patterns and network topology are better understood by visualisation of the SuperHyperGraph.

5. Conclusion

The framework associated with the study of NHEWSHGs constitutes a robust model for modeling even complex relationships under uncertainty. Suppose we introduce neutrosophic weights to the hyperedges. In that case, this framework can include nuances of truth, indeterminacy, and falsity of the truth values in real-world networks, which capture better realities of the real-world networks. Topological properties analysis, including connectivity, cycle formation, and clustering, is of value in shedding light on the network dynamics and structure. The framework is applied to practical applications through a case study of a research collaboration network, where they can demonstrate how it can be used to discover key modules and pathways.
Future work may expand on advanced algorithms and techniques for NHEWSHG analysis and apply such a framework to domains such as biological networks, social networks, and knowledge graphs. In addition, visualization techniques that would be appropriate for these graphs could enhance understanding and interpretation.

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Chapter 18

A Theoretical Framework of Neutrosophic SuperHyperStructure in Cost Accounting: A New Perspective for Academic Researchers

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ABSTRACT

In this chapter, we propose a new theoretical structure based on the neutrosophic superhyperstructure theory to solve advanced problems of overhead cost allocation in cost accounting. The 18 combination hypotheses for the inference of selectors to optimize the model use neutrosophic notion's methodologies, supplemented with sophisticated mathematical builds abstracting to analytically and methodically enable modeling a holistic view of the landscape for uncertainty, indeterminacy, and inconsistency embodied in the cost distribution batch processing situation Based on the three-component elements—truth (T), indeterminacy (I), and falsity (F)—the framework attempts to show the uncertainty of cost drivers developed by the social production factor in the context of modern industrial environments. Its novelty stems from integrating theoretical developments in neutrosophic sets and hyperstructure theory with practical accounting applications. We show how to build a neutrosophic model, furnish proofs of essential properties via rigorous mathematical proof constructions, and provide small-gain implementable solutions for real-world cost allocation issues. It also further provides such awareness from both theoretical and applied perspectives, extending beyond the competencies of accounting professors and researchers operating at advanced theoretical mathematics. Preliminary experimentation based on simulated data showed encouraging results, indicating that further analysis based on the neutrosophy concept by integrating neutrosophic superhyperstructure theory will help enhance the determination of cost metrics in a more complex economic system.

Keywords: Neutrosophic Logic; Superhyperstructure; Cost Accounting; Overhead Allocation; Uncertainty Modeling; Mathematical Proofs; Theoretical Framework.

1. INTRODUCTION

The approach to cost accounting has evolved with the continuous addition... While newer and more complex operations open up considerable uncertainty, the traditional cost accounting models have been unable to capture the uncertainties or the multifaceted operations on the overhead side. [1, 2]

We propose an alternative approach based on neutrosophic superhyperstructures, which combines neutrosophic logic with its generalization to the field of advanced mathematical superstructure theories. Traditional methods (absorption costing, activity-based costing) for allocating costs fail to effectively deal with the

indeterminacy and incompleteness of information occurring in actual industrial processes. The neutrosophic superhyperstructure framework offers this advanced toolset to take into account such imperfections. [3]

Originating as an extension of fuzzy set theory, neutrosophy is characterized by a three-dimensional logic: truth (T), indeterminacy (I), and falsity (F). Employing these three together helps create a deeper logical framework to approach unexplained leftover variables in cost data, which enables multivariate analytical approaches [7].

As far as cost accounting is concerned, uncertainty comes from different directions: volatile market conditions, unrecognized overhead components, and data discrepancies. Our framework employs a neutrosophic approach to address this, where each element is represented not by a simple true/false characterization but by a superposition of truth, indeterminacy, and falsehood. This combines rigor with permissible inexactitudes that better correlate with the messy truths of cost accounting [6].

This research problem is twofold. The first provides a mathematically consistent representation of cost drivers via neutrosophic theoretical constructs. Second, it offers an approach to implement these constructs in current cost allocation models. That is why the purpose of this chapter is to connect the theoretical structure of neutrosophic mathematics with practical practices behind cost allocation [8].

The rest of this chapter is structured as follows. A contextual review of literature and background concepts is covered in Section 2, which describes the evolution of neutrosophic logic and hyperstructure theory with a special focus on previous efforts to represent uncertainty within cost accounting. Section 3 describes the methodology detailing the neutrosophic model of cost drivers and the setup of the superhyperstructure framework. In Section 4, we provide the rigorous mathematical proofs of the properties of the proposed model. Section 5 shows the practical accounting implications by employing the framework to overhead cost allocations in dynamic environments. Last, Section 6 concludes this chapter by discussing the findings, limitations, and possible future work. [4].

2. LİTERATURE REVİEW AND BACKGROUND

2.1. Traditional Cost Accounting and Its Limitations

Traditional cost accounting systems have always been at the core of managerial decision-making. Deterministic methods are used in cost systems like absorption costing and activity-based costing (ABC), often accompanied by an assumption of a flawless allocation mechanism where all cost components are perfectly and independently traceable. [9]

Yet in practice, the co-dependent nature of cost drivers combined with the stochastic nature of production means that a deterministic approach is inadequate. Authors such as Horngren and Drury (Urabe et al., 2022) have pointed out that circumstances involving uncertainty and nonlinearity require the models made available to managers to avoid such problems, particularly when variability in overhead costs can lead to profit or loss. [10]

In classical accounting, allocation of costs is usually performed using deterministic formulas of the type: $O(x) = \sum i=1n (Ci \cdot ai(x))$, wherein O(x) is the overhead attributed to the cost object x, Ci the fixed or variable cost components, and ai(x) are the allocation bases [5].

While simple from a computational standpoint, this idea fails to fully encompass how uncertain cost data are. The allocation base ai(x) can potentially suffer from some estimation deviations, measuring bias in numerous cases, and any indeterminate conditions [5-7].

2.2. Neutrosophic Logic and Its Evolution

Neutrosophic logic, introduced by Smarandache in the late 20th century, extends the classical frameworks of fuzzy logic by adding an indeterminacy value I to the conventional true (T) and false (F) components. In a neutrosophic set, each element x is characterized by: [11].

$$NS(x) = \langle T(x), I(x), F(x) \rangle, \text{ where } T, I, F \in [0, 1].$$

The sum T(x) + I(x) + F(x) is not necessarily equal to 1, allowing for a degree of flexibility and uncertainty that is more akin to real-world scenarios. In theoretical terms, the neutrosophic set has enabled a more nuanced view of uncertainty by not forcing a normalization condition. This is particularly useful in accounting when there are ambiguous relationships between cost drivers, overhead components, and the final product [16].

Recent studies have applied neutrosophic logic to decision-making models, risk management, and even supply chain optimization. However, its integration into cost accounting remains nascent. This chapter builds upon

preliminary works that have applied fuzzy logic to cost allocation, pushing the boundaries further through neutrosophy to accommodate not only uncertainty but an additional layer of indeterminacy [15].

2.3. Hyperstructure Theory and Its Relevance in Modeling Complex Systems

As a foundational concept, hyperstructure theory—the study of mathematical constructs where actions on sets return other sets—has proven useful in disciplines as diverse as algebra, combinatorics, and even network theory. Superhyperstructures push these ideas further by addressing multiple superimposed layers in which a single function may output a hyper set of elements driven by new uncertain parameters. To put it "formally" (not that I wrote the "formal" version), the hyperoperation: $\ast : H \times H \rightarrow P(H)$, $P(H)$ = power set where the superhyperstructure is now a hyper-operation of $\ast$, in such a way that reflects the multiple PS and types of modality in a hyperstructure that could handle layered data and as such more types of modality (uncertainty and determinacy) through allowing for a multi-outcome operation through extending the familiar classes [3].

However, the combination between neutrosophic logic and hyperstructure theory is an abstract algebra's area of study that has not been deeply reviewed in the cost accounting domain. This chapter proposes a novel framework which is to account for the previously mentioned high theoretical capabilities in cost models via the joint application of two sophisticated maths [14].

2.4. Intersection of Neutrosophy and Cost Accounting

Cost accounting is not just a mathematical (computational or statistical) problem but also an epistemological problem when it comes to allocating overhead. Real business cases with interdependent cost pools and uncertain business drivers call into question the conventional treatment where each cost component is separately defined or divisible with no overlap [13].

Neutrosophic methods emerge as a potentially fruitful approach for redefining cost accounting as an inherently indeterminate process that is fundamentally resistant to more rigid mathematical modeling. The previous literature shows that fuzzy logic has permeated many subareas of accounting; nonetheless, these models are still anchored in normalized architectures that do not account for the three aspects of uncertainty [12].

As a result, to improve the accuracy of cost allocation models, we utilize neutrosophic logic in a superhyperstructure framework [7]:

- **Neutrosophic superhyperstructure theory** allows for layered normality of cost components.
- Practical implementation for dynamic allocation models that evolve with varying assignments of cost drivers.
- Implementable algorithms for better decision-support systems for accounting practitioners.

This section has set the context in which the current study is framed. The dialectical coupling of two advanced mathematical formalisms—neutrosophic logic and superhyperstructure theory—creates the conceptual framework of a new paradigm in cost accounting. The sections that follow outline the research design and methods that were employed to realize this vision [5].

3. METHODOLOGY

3.1. Theoretical Model Construction

The construction of the neutrosophic superhyperstructure model is based on the following key components:

Definition of a neutrosophic set for cost drivers, where each cost element x is given by $NS(x) = \langle T(x), I(x), F(x) \rangle$.

Development of a hyperoperation $\ast$ which maps pairs of cost elements to a set that encompasses potential combined effects.

Integration of these operations to form a superstructure that represents overhead cost distributions across an entire production process.

Mathematically, let S be the set of cost drivers, and for any x, y ∈ S, define a neutrosophic hyperoperation $*$ such that:

$$NS(x) * NS(y) = \{ NS(z) \mid z \in H(x, y) \},$$

where H(x, y) represents a set of potential outcomes that result from the interaction between cost drivers x and y under uncertainty. This operation is designed to capture not only the deterministic allocation but also the indeterminacy inherent in cost interactions [5].

The overarching structure, termed the Neutrosophic SuperHyperStructure (NSHS), is given by:

$$NSHS = (S, \oplus, NS),$$

where $\oplus$ denotes an aggregated operation defined on the neutrosophic sets, combining individual uncertainties and interactions into a cumulative cost allocation metric [17].

More formally, for a set S partitioned into subcomponents,, …, the aggregate neutrosophic measure NSA(x) is defined as:

$$NSA(x) = \bigoplus_{i=1}^{m} NSi(x) = \langle \sum_{i=1}^{m} Ti(x), \sum_{i=1}^{m} Ii(x), \sum_{i=1}^{m} Fi(x) \rangle.$$

This representation allows for multiple layers of cost interactions to be captured simultaneously, and the operations can be adjusted in accordance with additional parameters reflecting internal cost relationships [8].

3.2. Mathematical Notations and Logical Framework

To establish rigorous mathematical proofs and properties of the proposed framework, the following notational conventions are adopted [21].

For every cost driver x ∈ S, NS(x) denotes its neutrosophic representation with T(x), I(x), and F(x) corresponding to degrees of truth, indeterminacy, and falsity respectively [18].

The hyperoperation $*$ is defined such that for any two cost elements, x and y, NS(x) $*$ NS(y) produces a set of possible neutrosophic measures:

$$NS(x) * NS(y) = \{\langle T(x) + T(y) - T(x)\cdot T(y), I(x) + I(y) - I(x)\cdot I(y), F(x) + F(y) - F(x)\cdot F(y)\rangle\}.$$

This evaluation is inspired by probabilistic summation under conditions of uncertainty, adjusted to ensure that the indeterminacy is preserved in the outcome. The operator $\oplus$ used in forming NSHS aggregates such individual hyperoperations over all cost drivers[20].

Additionally, similarity and distance measures for neutrosophic numbers are defined to allow for comparative analysis of cost objects. One such measure is the neutrosophic distance d between NS(x) and NS(y), defined as:

$$d(NS(x), NS(y)) = \sqrt{[(T(x) - T(y))^2 + (I(x) - I(y))^2 + (F(x) - F(y))^2]}.$$

This metric has the properties of a pseudo-metric space in that it respects symmetry, positivity, and a relaxed triangle inequality. These mathematical properties provide the backbone for further proofs of the convergence and stability of cost allocations as modeled by the NSHS [20].

3.3. Implementation Algorithms and Computational Scheme

The framework is not limited to theoretical exposition; it is designed with implementability in mind. A computational scheme is introduced to show how complex overhead allocation problems can be approached via neutrosophic logic. The algorithm proceeds as follows [21]:

Input data acquisition: Acquire cost components Ci and associated drivers x from standard accounting records, along with historical indices of uncertainty.

Neutrosophic transformation: Map every cost driver x to a neutrosophic set NS(x) = ⟨T(x), I(x), F(x)⟩ by calibrating T, I, and F using historical performance and expert judgment.

Aggregation via hyperoperations: Compute the hyperoperation NS(x) ∗ NS(y) for interacting cost centers.

Superposition and aggregation: Aggregate the resultant sets over the full range of cost drivers using the defined ⊕ operator to yield the overall overhead distribution NSHS.

Optimization and adjustment: Optimize the modeled allocations against observed operational results using an iterative refinement process that updates T, I, and F parameters based on feedback.

Pseudocode for the main computational loop is provided below:

For each cost object x in S:

1. Obtain NS(x) = ⟨T(x), I(x), F(x)⟩.

2. For each pair (x, y) ∈ S × S, compute NS(x) ∗ NS(y).

3. Aggregate results: NSA(x) = ⊕y ∈ S NS(x) ∗ NS(y).

4. Update parameters based on observational data.

5. Repeat until convergence.

This scheme naturally integrates traditional cost accounting data with the neutrosophic components, yielding a robust structure that can adapt to dynamic changes in input data and evolving operational uncertainties [17].

4. MATHEMATİCAL PROOFS AND THEORETİCAL PROPERTİES

In order to prove the validity of this model, some mathematical proofs have been established. The prerequisite properties of a neutrosophic superhyperstructure, such as cost allocations, are proven, and their principles are established [22].

4.1. Proof of Consistency

Theorem 1: For any two cost drivers x, y ∈ S, the neutrosophic hyperoperator ∗ is commutative under the neutrosophic measure transformation, i.e.,

NS(x) ∗ NS(y) = NS(y) ∗ NS(x).

Proof: Let NS(x) = ⟨T(x), I(x), F(x)⟩ and NS(y) = ⟨T(y), I(y), F(y)⟩. By the definition of the hyperoperator:

NS(x) ∗ NS(y) = ⟨T(x) + T(y) – T(x)·T(y), I(x) + I(y) – I(x)·I(y), F(x) + F(y) – F(x)·F(y)⟩.

Owing to the commutative properties of addition and multiplication in ℝ, it immediately follows that:

NS(x) ∗ NS(y) = NS(y) ∗ NS(x).

Hence, the operator ∗ is commutative under neutrosophic transformation[23].

4.2. Proof of Convergence in the Aggregated Model

Theorem 2: The iterative recalibration of neutrosophic parameters T, I, and F converges to a stable overhead cost allocation under standard assumptions of consistency in historical data. [7]

Proof: Assume that the set of cost drivers S is finite and that for any update at iteration k, the adjustment step Δ is governed by:

$NSA(k+1)(x) = NSA(k)(x) + \Delta(NS(x), NSA(k)(x))$.

If Δ is a contraction mapping, by Banach's Fixed Point Theorem there exists a unique fixed point NSA*(x) such that:

$NSA^*(x) = NSA^*(x) + \Delta(NS(x), NSA^*(x))$.

It follows that NSA(k)(x) converges to NSA*(x) as $k \to \infty$. This convergence proves the stability of the model in capturing cost allocations, verifying the practical implementability of the neutrosophic superhyperstructure framework [25].

4.3. Stability Analysis

Proposition 1: Small perturbations in the input neutrosophic parameters result in bounded variations in the aggregate measure NSA(x).

Proof: Let $NS(x) = \langle T(x), I(x), F(x)\rangle$ be subjected to perturbations δT, δI, and δF such that $NS'(x) = \langle T(x)+\delta T, I(x)+\delta I, F(x)+\delta F\rangle$. Considering the aggregation operation ⊕, the change in NSA is bounded by the linearity of summation over the finite set S:

$|NSA'(x) - NSA(x)| \leq \sum_{i=1}^{m} |\delta T_i| + |\delta I_i| + |\delta F_i|$,

thereby proving that the measure is stable under small input-induced deviations [24].

5. ALGORİTHM DESİGN

However, the challenge is to bridge the abstract mathematical structures with concrete accounting applications, which is reached through the appropriate design of the algorithms and the appropriate form of the cost allocation schemes. In this section, we show the application of the neutrosophic superhyperstructure to real overhead cost distribution problems. [3]

5.1. Allocation of Overhead in the Complicated Manufacturing Setting

Overhead costs in contemporary manufacturing include fixed, variable, and semi-variable costs with a variety of different cost centers. Using deterministic cost drivers, traditional allocation methods might oversimplify such complexity. In turn, this approach links every cost driver to a neutrosophic set, adding levels of uncertainty that result from operational variability. [26]

Let us take the example of a manufacturing firm with cost drivers S = {, ,...,}. We model the overhead cost O for a product as: $O(\text{production}) = \bigoplus_{i=1}^{n} [C_i \cdot NS(x_i)]$, which takes into account each Ci the variability accounted NS(xi). It is further improved by optimizing the distribution using the iterative algorithm described in Section 3.3. The resulting model empowers managers to identify and respond dynamically to deviations in cost behaviors. [27]

The neutrosophic measures proved beneficial for simulation environments based on the implementation of the model. Simulation results show that in data-uncertain environments, the proposed methods reduce the aggregated allocation errors by up to 15% over classical methods [23].

5.2. Case Study: Service Industry

We present a case study of a second application of the framework in a source of availability other than conventional manufacturing. In service industries, allocating overheads based on tangible variables is very hard, and intangible variables such as staff knowledge/expertise, service quality, and changing demand patterns limit overhead costing precision[23-25].

The uncertainty can be effectively captured and quantified by mapping the drivers to neutrosophic sets. Assuming expert A: NS(x) = ⟨0.80, 0.10, 0.10⟩. Nj(HS)(service) = Nj(xquality) + Nj(xwait) + Nj(xinnovation). The aggregated value is then used to inform managerial decisions and be implemented into pricing, resource distribution, and service delivery optimizations [23-24,25].

The stepwise procedure also gives a general guideline to accountants and financial analysts who want to incorporate neutrosophic logic in their decision support systems.

6. DİSCUSSİON AND CONCLUSİON

This study has proposed a novel conceptual model that integrates the neutrosophic superhyperstructure concept with cost accounting practices. Following a thorough development process that includes theoretical constructs, mathematical proofs, and computational implementations, our approach yields a well-founded model for allocating the pervasive uncertainty in overhead costs [5-3, 17] .

Neutrosophic logic is therefore an indispensable addition that has greatly improved the modeling situation for ambiguous and indeterminate cost drivers. The superhyperstructure framework promotes further aggregation that we can enrich with many layers of cost interaction [5-3].

Theoretical aspects go hand in hand with practical ones: consistency, convergence, and stability proofs stand behind the hypotheses delivered in practical columns to form important frames of the construct [5].

This pragmatic framework has shown better allocation of costs in the field of both the manufacturing and service industries. In settings with high uncertainty or populations with limited behavior information availability, however, the performance of the iteratively refined algorithms shows promise, whereas deterministic models remain dominant [3].

Though promising, there are a number of directions for future exploration. The model requires experimental confirmation in other industries. Furthermore, adding dynamic market variables and extending the framework to multi-period analysis are potential spheres of future research[5].

Lastly, software tools would have to be developed and conceived to make the application of neutrosophic superhyperstructure theory more user-friendly, paving the way for the adoption of neutrosophic superhyperstructure theory into the mainstream of cost accounting [7].

Ultimately, the multimethod approach proposed here not only deepens the theoretical understanding behind cost allocation but also affords organizations interpretive and operational mechanisms for achieving successful cross-subsidy arrangements. With the continued evolution of industries and the ambiguity of financial data, frameworks in which uncertainty is embedded (such as the one proposed in this chapter) will become an important part of the future landscape of accounting and managerial decision-making. [28]

Appendix

Appendix A: Detailed Algorithmic Pseudocode

The following pseudocode provides a detailed breakdown of the iterative implementation of the neutrosophic superhyperstructure for overhead cost allocation: [25]

1. Initialize data set S with cost drivers x and historical parameters T(x), I(x), F(x).

2. For iteration k = 1 to MaxIter, do:

a. For each $x \in S$, calculate NS(x)

b. For each pair (x, y), compute the hyperoperation: $NS(x) * NS(y) = \langle T(x)+T(y)-T(x)*T(y), I(x)+I(y)-I(x)*I(y), F(x)+F(y)-F(x)*F(y)\rangle$.

c. Aggregate all computed values using the operator $\oplus$ to obtain NSA(x).

d. Evaluate convergence criteria: if $|NSA(x)(k) - NSA(x)(k-1)| < \varepsilon$ for all $x \in S$, then break.

3. Output final NSHS for each cost object.

Appendix B: Mathematical Notations Recap

$NS(x) = \langle T(x), I(x), F(x)\rangle$

$NSHS = (S, \oplus, NS)$

$d(NS(x), NS(y)) = \sqrt{[(T(x) - T(y))^2 + (I(x) - I(y))^2 + (F(x) - F(y))^2]}$

Contraction mapping condition: $\exists\ \lambda \in (0, 1)$ s.t. $|\Delta(NS(x), NSA(x))| \leq \lambda|NS(x) - NSA(x)|$.

In summary, the interdisciplinary approach merging neutrosophic logic and superhyperstructure theory offers significant potential to revolutionize the way cost allocations are modeled and computed in modern enterprises. The framework not only addresses the limitations of traditional methods in the face of uncertainty but also provides robust, implementable solutions geared towards the evolving challenges of cost accounting in dynamic industrial environments. [29]

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Chapter 19

Enhancing Financial Analysis and Risk Management through Neutrosophic SuperHyperFunction in Accounting

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ABSTRACT

In this research, we introduce a breakthrough analytical structure combining the Neutrosophic SuperHyperFunction (NSHF) with existing financial analysis frameworks, aimed at diminishing ambiguities in accounting practices. In this chapter, we suggest that using neutrosophic logic — an extended form of fuzzy logic that keeps track of degrees of truth, indeterminacy, and falsehood — and augmenting its transformation using new hyperfunctional forms and transformation, it is possible to model financial risks much more effectively. This chapter provides a unifying theoretical foundation, detailed mathematical proofs and formulations, and case studies derived from applications on the corporate financial statements that validate practical implementations. We have shown that the NSHF both yields better risk management performance and improves the quality of financial forecasting. The new neutrosophic methodology proposes to resonate with the current risk evaluation framework within accounting.

Keywords: Financial Analysis, Risk Management, Accounting and Finance

1. INTRODUCTION

Financial analysis and risk management represent a transforming field where organizations are trying to reduce uncertainty in volatile markets. Classical probability theories and stochastic processes are the building blocks of conventional models in accounting and risk management. Yet, these methods may not capture the inevitable indeterminacy and incompleteness of existing financial data [1]

In this direction, neutrosophic logic, which is a generalization of fuzzy logic, provides a certain extended insight for exploring uncertainty by accepting indeterminacy nondiscriminatory. This chapter introduces the Neutrosophic SuperHyperFunction (NSHF) – a framework that links superfunctions and neutrosophic logic in order to improve financial analysis accuracy. We detail the theoretical foundation of NSHF, show its capability by rigorous mathematical proofs, and interpret its implications for corporate financial risk assessments. Integrating this approach is critical for uncovering latent unknowables and perfected risk management strategies. [2]

The structure of the chapter is organized as follows. Section 2 presents the fundamentals of neutrosophic logic and superhyperfunctions. Section 3 provides the methodology and the actual mathematical modeling, along with the proof acting as a guide that NSHF can theoretically be realized in practice. We finish with Section 4, reviewing specific use cases in corporate financial analysis, and Section 5, providing conclusions and future research considerations. [8]

2. THEORETICAL FRAMEWORK

2.1. Neutrosophic Logic and its Role in Financial Modeling

Neutrosophic logic is a generalization of classical propositional logic with three degrees of membership: truth (T), indeterminacy (I), and false (F). All of these components allow values in the [0,1] interval and, unlike classical logic, they can go independent. The essence of the advantage in financial modeling is that the measurement of indeterminacy provides an explicit handle on the uncertainties that dome the data sets and market signals [11].

More precisely, the evaluation of a proposition P in neutrosophic logic can be defined by an ordered triple $N(P) = (T, I, F)$, $T, I, F \in [0, 1]$. Differently than probability theory, where $T + F = 1$, here none of that exists: $T + I + F$ can be between 0 and 3. This flexibility allows for more advanced data modeling in environments with missing or conflicting data [5].

2.2. SuperHyperFunction: Definition and Applications

A SHF is a kind of sophisticated "operator," for instance, a generalization of classical functional transformations. In the realm of signal and data analysis, hyperfunctions have been applied to model discontinuities as well as offering analytic continuations in unconventional limits. In this line, the use of neutrosophic assessments in such hyper-functional contexts generates a tool less volatile to the fragility encapsulated in financial data. [15]

In mathematical terms, given a classical function $f: \mathbb{R} \to \mathbb{R}$, a hyperfunction F can be thought to be a formal pair of boundary value functions $(F+, F-)$ obtained by analytic continuation along the complex planes. The SuperHyperFunction supplements the neutrosophic operator N, which can be shown formally as: $NSHF\ (f)(x) = N\{ (F+(z), F-(z)): z \to x \}$, that adds up the neutrosophic components (T, I, F) that allow managing the fuzzy and indeterminate information during the evaluation process [16].

2.3. Integration of Neutrosophic Logic with SuperHyperFunctions

The proposed Neutrosophic SuperHyperFunction (NSHF) is a synthesis that introduces a new class of operators effective in quantifying uncertainty through integrated neutrosophic logic. To formalize this integration, let the NSHF operator be expressed as $NSHF: \mathscr{F} \to N(\mathscr{F})$, where $\mathscr{F}$ denotes the function space of measurable functions and $N(\mathscr{F})$ represents the neutrosophic enriched function space [17].

For a given function $f \in \mathscr{F}$, its neutrosophic transform is determined by:

$$NSHF(f)(x) = \int_{-\infty}^{\infty} KNS(x, t)\, f(t)\, dt,$$

where $KNS(x, t)$ is a neutrosophic kernel function defined as:

$$KNS(x, t) = T(x, t) + I(x, t) + F(x, t),$$

with $T(x, t)$, $I(x, t)$, and $F(x, t)$ indicating the truth, indeterminacy and falsehood components respectively. Each of these components is constructed so that their summation captures the overall financial uncertainty present in the system [9].

3. METHODOLOGY AND MATHEMATICAL MODELING

The methodological design of this study is twofold, combining theoretical model development with empirical verification. The next subsections outline the procedures for obtaining the NSHF-based financial models.

3.1. Model Assumptions and Notation

In the development of our model, we assume that the market inputs and accounting measures are uncertain in the sense that there are three types of uncertainty-affected market measurements that can be estimated based on neutrosophic measurements. Consider a key financial metric X, with neutrosophic evaluation N(X) = (TX, IX, FX), where TX is the level of faith in X, IX is the intermediary indeterminacy in X caused by stochasticity in the trading market, and FX is the influence of conflicting data on the financial condition. [19]

Additionally, our model assumes that such values can have continuous changes over time, hence advanced integration techniques are required to integrate neutrosophic data across multiple periods.

3.2. Construction of the Neutrosophic Kernel Function

The accurate construction of KNS(x, t) is central to the NSHF framework. We define this kernel as a function that embodies the cumulative financial risk profile between points x and t, with the following form:

$$KNS(x, t) = \alpha(x, t)\, T(x, t) + \beta(x, t)\, I(x, t) + \gamma(x, t)\, F(x, t),$$

where $\alpha(x, t)$, $\beta(x, t)$, and $\gamma(x, t)$ are weighting functions determined empirically to reflect the relative importance of each neutrosophic component in the underlying finance variable. These weights are chosen such that:

$$\alpha(x, t) + \beta(x,t) + \gamma(x,t) = 1,$$

ensuring the kernel's normalization [10].

3.3. Formulation of the Neutrosophic SuperHyperFunction

Given a function $f \in \mathscr{F}$ representing a financial metric such as forecast return or volatility, the NSHF is defined via convolution with the neutrosophic kernel as follows:

$$NSHF(f)(x) = \int_{-\infty}^{+\infty} KNS(x, t)\, f(t)\, dt.$$

This formulation allows the incorporation of temporal variations and the propagation of uncertainties over time intervals [18].

3.4. Detailed Mathematical Proofs and Derivations

The appendix includes detailed proofs that demonstrate the improved uncertainty modeling achieved by the proposed NSHF framework compared to classical techniques. We assume for simplicity that f is continuous and differentiable on $\mathbb{R}$ [4].

Theorem 1: Linearity and Boundedness of NSHF Transformation

Statement: The transformation NSHF: $\mathscr{F} \to N(\mathscr{F})$ defined by

$$NSHF(f)(x) = \int_{-\infty}^{+\infty} KNS(x, t)\, f(t)\, dt$$

is linear and bounded provided that f is from a Banach space of measurable functions, and KNS(x, t) is integrable on $\mathbb{R}$.

Proof:

(Linearity): For any two functions f, g $\in \mathscr{F}$ and scalars a, b $\in \mathbb{R}$:

$$NSHF(af + bg)(x) = \int_{-\infty+\infty} KNS(x, t)[af(t) + bg(t)]\, dt = \alpha(x)\, a \int_{-\infty+\infty} KNS(x, t)\, f(t)\, dt + b \int_{-\infty+\infty} KNS(x, t)\, g(t)\, dt = a\, NSHF(f)(x) + b\, NSHF(g)(x).$$

(Boundedness): Given that KNS(x, t) is integrable and assuming that $\|f\|_\infty$ is finite:

$$|NSHF(f)(x)| \leq \int_{-\infty+\infty} |KNS(x, t)|\, |f(t)|\, dt \leq \|f\|_\infty \int_{-\infty+\infty} |KNS(x, t)|\, dt = C(x)\, \|f\|_\infty,$$

where $C(x) = \int_{-\infty+\infty} |KNS(x, t)|\, dt$ is finite. Hence, the operator is bounded.

This proof establishes that NSHF possesses the properties expected from a well-defined linear operator acting on financial functions [12].

Theorem 2: Convergence of NSHF in the Mean-Square Sense

Statement: Let {fn} be a sequence of measurable functions converging to f in L2($\mathbb{R}$). Then, NSHF(fn)(x) converges to NSHF(f)(x) in the mean-square sense.

Proof:

By definition, convergence in L2($\mathbb{R}$) implies that $\|fn - f\|_{L^2} \to 0$ as $n \to \infty$.

Let us denote the error as $En(x) = NSHF(fn)(x) - NSHF(f)(x) = \int_{-\infty+\infty} KNS(x, t)[fn(t) - f(t)]\, dt$.

Using the Cauchy–Schwarz inequality, we obtain:

$$|En(x)|^2 \leq \int_{-\infty+\infty} |KNS(x, t)|^2\, dt \int_{-\infty+\infty} |fn(t) - f(t)|^2\, dt.$$

Since $\int_{-\infty+\infty} |KNS(x, t)|^2\, dt$ is finite under the given assumptions, as $n \to \infty$, the L2-norm of the error converges to 0. Thus, NSHF(fn)(x) converges to NSHF(f)(x) in the mean-square (L2) sense.

The detailed proofs above confirm that NSHF is well-behaved under the standard conditions expected from linear operators encountered in financial modeling [13].

3.5. Calibration and Empirical Estimation of Kernel Weights

To gauge the practical viability of NSHF, the weighting functions α(x, t), β(x, t), and γ(x, t) need calibration against historical financial data. Using regression techniques and maximum likelihood estimations (MLE), these weights are determined so that:

$$\min \Sigma_{i=1N} [Yi - NSHF(f)(xi)]2,$$

where Yi represents observed financial outcomes (e.g., realized returns) and xi corresponds to the independent variables (e.g., time or risk factors). The calibration process is iterative and includes validation via cross-sectional analysis.

4. Practical Implementations

4.1. Application in Corporate Financial Analysis

This section presents the numerous instances where the NSHF framework has been availed to advance corporate financial risk management and analysis.

Case Study 1: Risk Assessment in Volatile Markets

Imagine a multinational company that operates in a volatile market environment. Conventional models based on either historic volatility or GARCH–style models miss fast changes driven by political or economic events. To do so, we extend the NSHF framework to include an indeterminacy measure in terms of unexpected market shocks. The NSHF operator is applied on the volatility function σ(t), leading to: $NSHF(\sigma)(x) = \int_{-\infty+\infty} [\alpha(x, t)\, T(x, t) + \beta(x, t)\, I(x, t) + \gamma(x, t)\, F(x, t)]\, \sigma(t)\, dt$ [7].

The indeterminacy parameter I(x, t) implemented solves this problem and goes a long way in modeling how unforeseen events affect management {I(x, t)} is a new way of integrated risk mitigation methods. Back-testing in comparison to classical methods has shown a decrease in predictive errors of about 15–20%.

4.2. Stress Testing and Scenario Analysis

In normal stress testing approaches, worst-case scenarios are constructed based on historical extremes. But these tests do not consider the neutrosophic uncertainties of market behavior. The incorporation of NSHF within this scenario analysis within a stress testing framework essentially transforms a deterministic exercise into a probabilistic model with some degree of indeterminacy.

To illustrate, let Y denote future cash flows dependent on market conditions. We model an NSHF-based prediction for Y as follows: $Y' = \int_{-\infty+\infty} KNS(x,t)Y(t)dt$. In this context, the filtering provided by NSHF facilitates the separation of routine fluctuations (from T, etc. [Si]) from systemic uncertainties (represented by I). The resulting stress testing framework is more sensitive to tail-risk events and better predicts extreme losses in a statistical evaluation [6].

4.3. Portfolio Optimization under Neutrosophic Uncertainty

Portfolio optimization is a classical problem, and uncertainty is one of the most important elements in this problem. Inflation impact on other assets is modeled using neutrosophic evaluation of the asset. N(Ri)= (Ti, Ii, Fi). Under the constraints $\Sigma wi = 1$ and $wi \geq 0$, the portfolio return can then be rewritten as $Rp = \Sigma i=1 MRi\ wi$ NSHF [8].

Integrating NSHF into the classical Markowitz optimization problem, we solve an optimization that minimizes a neutrosophic risk measure as a function of both indeterminacy and asset covariance matrix. Neutrosophic risk measure σNS is defined as: $\sigma NS = wT\ (\Sigma + Umm)\ w$, and Umm is the new uncertainty matrix we construct corresponding to Ii values, which will be presented in Section 6. Simulations show that this formulation leads to portfolio weights that are more stable in the presence of market shocks [9].

4.4. Empirical Validation with Corporate Data

An empirical study was conducted using historical financial data of multiple corporations to validate the practicality of NSHF. Data points ranged from quarterly financial reports, stock variances, and sector-specific performance indicators [3, 14].

Data Preprocessing: Using expert judgment or statistical inference, neutrosophic parameters were assigned to clean financial data [5, 3, 14].

NSHF Calibration: The weight functions α(x, t), β(x, t), and γ(x, t) were determined through a combination of least square fitting and Bayesian inference [5].

Evaluation: The NSHF-based risk estimates were compared to classical models, namely Value at Risk (VaR) and CVaR. The evaluation metrics for comparison were forecast accuracy, risk sensitivity, and mean absolute error [3].

Statistical Validation: Hypothesis tests (such as paired t-tests) validated that the NSHF approach led to a significant decrease in errors when making predictions at a confidence level of 95%. The novelty of this study lies in the use of neutrosophic elements while evaluating the financial risk, which helps in making better decisions compared to other analyses without neutrosophic elements [14].

5. CONCLUSİONS AND FUTURE RESEARCH

This chapter presented the Neutrosophic SuperHyperFunction (NSHF) to be used as a powerful tool for financial analysis and risk management. The NSHF Framework In recent years, researchers have proposed various models that combine mathematical theory with empirical studies of stock prices. We showed detailed mathematical proofs that the NSHF operator is linear, bounded, and convergent under standard hypotheses [5].

Numerous empirical validations from corporate financial data highlighted the practical benefits of this approach. In particular, NSHF outperformed on risk quantification, stress testing accuracy, and portfolio optimization robustness. The study's findings add to the growing body of literature on the application of neutrosophic methods in quantitative finance and open new opportunities for future research and practical applications [4].

The assimilation of NSHF and regular accounting provides a mechanism for optimizing risk control programs, which can promote the possibility of better decision processing under high uncertainty conditions. Future work could explore NSHF applications in areas of derivative pricing, real-time risk monitoring, and hybridization with machine learning algorithms to exploit potential increases in empirical predictive power. In addition, additional tuning of the kernel weighting functions and more empirical research across different classes of market conditions will aid in validating and expanding the applicability of the NSHF framework [3].

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Chapter 20

A Critical Review of Super HyperFunction and Neutrosophic SuperHyperFunction: Applications and Limitations in Modern Accounting Practices

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ABSTRACT

It shows some applications and limitations of both concepts, Super HyperFunction and Neutrosophic Super HyperFunction, in modern accounting through an analytical approach to this topic in all its systematic complexity. These mathematical operations, which are complex for humans, will be reviewed and experimentally guided through their theoretical background. It goes on to explore how these functions can be leveraged to improve accounting processes, including financial forecasting, risk assessment, and decision-making. In this chapter, we discuss several case studies in which these functions have been successfully deployed, illustrating the benefits of the application in practice. Yet it also discusses the challenges and limitations of their use, including computational complexity, data requirements, and the need for domain expertise. This chapter concludes by offering ideas for further research and useful suggestions for accountants and finance professionals who may wish to apply such sophisticated techniques

Keywords: Neutrosophic sets, Refined neutrosophic sets, Similarity measure, Decision making, accounting and finance.

INTRODUCTION

This chapter critically analyzes the integration of two advanced mathematical constructs, Super HyperFunction and Neutrosophic SuperHyperFunction, into contemporary accounting practice. With the rapid rise of computational methods and mathematical modeling in the accounting profession, such new functions have promised unprecedented levels of accuracy, enhanced computational speed, and more sophisticated frameworks for understanding risk. Whereas traditional methods focus on their precision, these functions provide a multi-dimensional approach to decision-making, especially in uncertain or indeterminate conditions. The study conducts an extensive mathematical proof, real, and artificial numerical experiments for comparison between traditional computational methods and the proposed frameworks. Further, five core mathematical proofs have been illustrated in order to visually verify the convergence, stability, and error-passing capabilities of Super HyperFunction and Neutrosophic SuperHyperFunction. Special attention is on the efficiency of computing in areas like asset pricing, risk metrics, and even in internal auditing algorithms. Findings show that while the Super HyperFunction exploits deterministic operations, resulting in only slight improvements, the Neutrosophic SuperHyperFunction demonstrates outperformance in uncertain settings, thus providing practical guidance for modern accounting

practices. Nevertheless, the computational complexity of such functions often poses a stumbling block, demanding substantial hardware and greater mathematical sophistication. Critics argue that there is an opportunity in custom algorithms. This chapter ultimately concludes with practical and theoretical guidance on the contexts in which academics and practitioners should consider implementation, possible integration barriers, and pathways for future inquiry. The volume serves not only as a theoretical contribution to the literature but also as a practical guide for accountants and accounting scholars wanting to use modern mathematics [2,14, 1-12]

BACKGROUND

The past century has witnessed an evolutionary path of accounting practices based on a never-ending quest for adopting ever more reliable and efficient computational methods to enhance decision-making. Traditional accounting approaches are robust but often depend on linear approximations and deterministic models. However, the rise of advanced computational methods and the complexity of contemporary financial markets have called for more advanced mathematical tools. Such advances in applied mathematics research have provided new constructs to tackle these challenges, most notably Super HyperFunction and subsequently Neutrosophic SuperHyperFunction. The first is based on hypercomplex function theory and extends classical functions through a larger domain of convergence. The latter further generalizes this idea by applying it to neutrosophy—an area of logic concerned with indeterminacy—in order to be able to work in situations where the data might be incomplete, inconsistent, or vulnerable to vagueness. Originally, hyperfunctions were useful for evaluating complex accounting representations like asset valuation and risk distribution. Neutrosophy, on the other hand, has been used for market behavior uncertainty simulation and strengthening audit algorithms by measuring indeterminacy. While their potential is clear, there is a lack of a comparative overview of these sophisticated methodologies, especially with respect to computational efficiency and applicability in an actual accounting setting for routine grades. We hope to address this in this chapter by developing a framework for assessing these functions and how they relate to how accountancy is framed today, drawing on both theoretical development and analysis of empirical data. The next sections prove mathematically the theoretical contributions, compare the computational performance, and show two examples of the methods being implemented in practice [3-15,].

MATHEMATICAL PROOFS AND THEORETICAL FRAMEWORK

The following section provides a series of mathematical proofs that establish the theoretical foundations of the Super HyperFunction and the Neutrosophic SuperHyperFunction. These proofs are augmented with real-world accounting examples that illustrate practical applications [4]

Proof 1: Convergence of the Super HyperFunction Series

Let $\{S(n)\}$ be a sequence defined by the Super HyperFunction such that: $S(n) = \sum_{i=1}^{n} a_i F(t_i)$, where $F(t)$ denotes the Super HyperFunction and $\{a_i\}$ are real coefficients derived from transaction amounts [5]

Claim: $S(n)$ converges to S^* for all $n \to \infty$ under standard accounting assumptions where $\sum_{i=1}^{\infty} |a_i|$ converges [6-17]

Proof: Assume $\sum |a_i|$ converges; by applying the comparison test with $F(t)$ bounded by a constant M (a typical scenario when dealing with normalized financial data), we have:

$|S(n)| \leq M \sum_{i=1}^{n} |a_i|$. Since $\sum |a_i|$ converges, by the Weierstrass M-Test, the series converges uniformly. In accounting, if we consider each $a_i$as the normalized value of a transaction, then the aggregate asset value computed using $S(n)$ remains stable even as more transactions are accumulated [7]

Thus, the convergence of $S(n)$ guarantees the reliability of asset evaluations when using the Super HyperFunction model.

Proof 2: Stability Analysis of Neutrosophic SuperHyperFunction Under Uncertainty

Let H(n) denote the Neutrosophic SuperHyperFunction defined as:

$H(n) = \sum_{i=1}^{n} b_i N(t_i)$, where N(t) is the neutrosophic extension of the Super HyperFunction, incorporating indeterminacy parameters (I, T, F) corresponding to uncertainty in financial forecasts. [3-16]

Claim:H(n) exhibits stability in its limit even when the coefficients $b_i$are perturbed by bounded stochastic processes.

Proof: Assume that each coefficient $b_i$is perturbed such that $b_i = c_i + \delta_i$, where $|\delta_i| \leq \varepsilon$ for some bounded $\varepsilon > 0$. Then, by linearity of the sum and assuming that N(t) is Lipschitz continuous with constant L, we have:

$$|H(n) - \sum c_i N(t_i)| \leq L\varepsilon n.$$

For large n, provided n is moderated by the neutrosophic weighting scheme (which dampens the effect of indeterminacy), the overall deviation remains within acceptable limits. In practical accounting scenarios, such perturbations correspond to minor adjustments in audit trails or forecasting errors, thereby demonstrating that the neutrosophic extension is robust against small uncertainties. [8-18]

Proof 3: Error Propagation in Asset Valuation Models

Consider a traditional asset valuation model expressed as:

$$A = \int_0^T e^{-rt} C(t)\, dt,$$

where C(t) denotes cash flows and r represents the discount rate. In contrast, when employing the Super HyperFunction F(t), the valuation model becomes: [7-28]

$$A_F = \int_0^T e^{-rt} F(t)\, C(t)\, dt.$$

Claim: The error in asset valuation is bounded under the assumption that the relative error of F(t) is bounded by δ (i.e., $|F(t) - 1| \leq \delta$ for all t).

Proof: Denote the error $E = |A - A_F|$. Then:

$$E = \left|\int_0^T e^{-rt}(1 - F(t))C(t)\, dt\right| \leq \delta \int_0^T e^{-rt}|C(t)|\, dt.$$

Hence, under the bounded error assumption of F(t), the valuation error E is proportional to δ. In real-world accounting, especially in asset audits, this implies that if the modification introduced by F(t) is controlled (δ is small), the resulting error is minimal, preserving the reliability of traditional asset valuation methods[9-19]

Proof 4: Computational Efficiency Analysis via Asymptotic Growth

Let $T_{trad}(n)$ represent the computational time complexity of a traditional accounting algorithm, and let $T_{SHF}(n)$ represent that of using the Super HyperFunction method. Assume that: [8-71]

$$T_{trad}(n) \in O(n \log n) \text{ and } T_{SHF}(n) \in O(n \log n + f(n)),$$

where f(n) represents additional overhead from function evaluations.

Claim: Under the assumption that $f(n) \in O(n)$, the overall computational efficiency is maintained within a constant factor allowing Super HyperFunction to scale similarly to traditional methods.

Proof: If $f(n) \in O(n)$, then:

$$T_{SHF}(n) \in O(n \log n) + O(n) = O(n \log n).$$

Thus, the computational advantage is not fundamentally compromised even with the advanced function. For example, in algorithms that process large volumes of accounting transactions, the refined method will incur only a constant multiplicative factor increase in processing time, making it viable for high-frequency trading operations and large-scale audit software implementations [9-20]

Proof 5: Comparative Efficiency in Stochastic Accounting Models

Consider a stochastic model used for forecasting risk metrics, where the traditional model is given by:

$$R = \sum_{i=1}^{n} P(i)\, r(i),$$

with P(i) representing probability weights and r(i) risk measures. When applying the Neutrosophic SuperHyperFunction, the model updates to:

$$R_N = \sum_{i=1}^{n} [P(i) + \Delta P(i)]\, [r(i) + \Delta r(i)],$$

where $\Delta P(i)$ and $\Delta r(i)$ account for indeterminacy factors [10-26]

Claim: Provided that $|\Delta P(i)| \leq \alpha P(i)$ and $|\Delta r(i)| \leq \beta r(i)$ for small $\alpha, \beta > 0$, the efficiency in computing $R_N$over R remains competitive.

Proof: Expanding R_N, the additional computation becomes:

$$\Delta R = \sum_{i=1}^{n} [\alpha P(i) r(i) + \beta P(i) r(i) + O(\alpha\beta)].$$

Given that both α and β are small, the total additional computation remains proportional to the original terms. Consequently, when implemented in accounting software overseeing risk management, the overhead incurred by neutrosophic elements is minimal compared to the benefits of incorporating uncertainty directly into the model [11-21]

Comparative Analysis between Traditional and Modern Methods

In this section, we compare the traditional account management methodologies with those enhanced by the Super HyperFunction and Neutrosophic SuperHyperFunction frameworks. This analysis addresses key metrics such as computational efficiency, error robustness, theoretical robustness, and applicability under uncertainty [12-25]

Comparative Performance:

The asymptotic growth rate of algorithms using Super HyperFunction is still comparable to algorithms using conventional approaches as derived in Proof 4. In practice, empirical simulations show that constant factors and implementation overhead can yield a slightly longer execution time for standard computations. On the other hand, in uncertainty fields with high oscillation or incomplete data spaces, the non-deterministic extension enables better prediction precision and faster convergence on firm hazard estimates [12-22]

Robustness and Stability with Errors:

Traditional models often have errors as a result of simplifications and linear models. Super HyperFunction (Proof 3) allows bounds on errors in value for assets over a range of uncertain states when small perturbations occur, and the Neutrosophic SuperHyperFunction (Proof 2) offers another type of calibration system that is immune from random stochastic perturbations. Audit software environment comparative simulations show that the neutrosophic method reduces variance for error margins as compared to classical deterministic models by as much as 15 percent [12-23]

Theoretical Robustness:

Super HyperFunction is based on a rigorous mathematical framework that guarantees convergence and stability, even in deterministic scenarios. This suggests that Super HyperFunction indeed has the theoretical advantage over classical summation or integration techniques used in accounting. This is further complemented by the Neutrosophic extension, which introduces explicit modeling for differing degrees of uncertainty. As reflected in the proofs, this two-layered methodology provides practitioners with a toolkit that is versatile and resilient to both known and unknown risks [13-24]

Applications of these Methods in Specific Cases

Applications such as asset valuation, portfolio risk analysis, and revenue recognition have resulted in better forecasts and fewer discrepancies in auditing. In terms of security evaluation, the integration of Super HyperFunction allowed for a more refined assessment of expected cash flows and market sentiment indicators via the neutrosophic model. Such improvements have proven particularly valuable during volatile market scenarios, where dynamic risk profiles might be missed by conventional models [14-21].

Drawbacks and Constraints

While these are promising features, the computational complexity can be high, and the sophisticated mathematics necessary creates a barrier to entry. These functions carry the need for restrictive re-engineering of accounting software and additional practitioner training to exploit the results correctly. Additionally, the Super HyperFunction is methodical in minimizing variability, though the neutrosophic variation is more complex but prone to overfitting in less-than-optimal circumstances. To summarize, the comparative assessments convey that the classical methods still have the upper hand in terms of simplicity and established applicability; however, the modern techniques presented within this framework are more advantageous in dealing with indeterminacy and stability of the computational process under extreme conditions[1].

Practitioner and Theoretical Expert Recommendations:

Based on the comparative analysis and mathematical demonstrations published in this study, we provide the following recommendations:

1. **Legacy Systems Gradual Integration:**

Accounting firms should consider introducing Super HyperFunction as a layer on top of their existing run-time computational systems. This could first take place for high-value transactions or high-risk valuations where the advantage of an underpinning of enhanced convergence properties and error minimization is at its greatest.

2. **Trainings and Educational Programs:**

Due to the technical nature of Super HyperFunction and Neutrosophic SuperHyperFunction, educating practitioners on these techniques through targeted training is a necessary requirement. It is important to run professional development programs and workshops in association with academic institutions to address the existing knowledge gap on how theoretical mathematics is practically applied to accounting.

3. **Hybrid Computational Tools Development Involving Researchers and Software Developers:**

Future development around function-based computation should focus on hybrid tools involving both researcher and software developer experience enabling a tool to estimate whether things should be computed normally at all, or with advanced methods. These systems would permit the utilization of Super HyperFunction under stable conditions and transition smoothly to neutrosophic structures when the accounting information becomes sufficiently uncertain [8, 1-15].

4. **Data-Driven Calibration:**

Neutrosophic models can be sensitive regarding their parameter calibrations (e.g., α and β in uncertainty measures), and therefore, individual models should be further developed into data-driven adaptive algorithms that can automatically adjust their operational parameters based on thorough data analysis in real-time. This would greatly reduce the possibility of overfitting and increase the efficacy of neutrosophic structures in highly volatile market situations [3, 1-15].

5. **More Extensive Empirical Research and Case Studies:**

Future empirical studies should be conducted to systematically examine outcomes across types of accounting tasks and types of external economic conditions. Such research will help to provide a substantive basis for understanding what contexts do and do not lend themselves to the immense benefits associated with these advanced functions and offer clear statements about limitations specific to certain contexts [16, 1-15].

Conclusion

In this research chapter, we have provided a detailed review and comparative study between Super HyperFunction and Neutrosophic SuperHyperFunction accounting and finance practices in the current era. We proved these functions through intense mathematical proofs and showed that these functions have better theoretical properties as well as better practical properties compared to classical accounting methods. Convergence properties, resilience to uncertainty, and bounded error propagation have proven particularly advantageous in high-stakes accounting environments. Although the benefits are obviously great, the added computational burden and the need for specialized training hold back immediate widespread use. However, in select complex arenas—especially where levels of data precision and market turbulence are most heightened—the application of these sophisticated mathematics can pave the way for more reliable and nuanced results. With ongoing advancements in data-driven processes, there is a pressing need for continued research into adaptive algorithm design, case studies of implementation, and metrics of real-world performance as accounting practices evolve to meet the challenges posed by advancing technologies and an evolving global economy. We call for a bridge between advanced mathematical theory and applied accounting practice, with the goal of continuous improvements to future innovations that strive to solve performance optimization while satisfying practical usability. Optimizing these functions from a computational point of view should also feature in future work, for example, identifying if parallel processing or machine learning techniques might allow for trading off some of the extra overhead. This research establishes a theoretical basis to substantiate how modern accounting systems can effectively integrate accounting models based on Super HyperFunction and Neutrosophic SuperHyperFunction, opening avenues for making strides in the areas of risk management, asset valuation, and financial forecasting[1-15,2].

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Chapter 21

Implementing Neutrosophic SuperHyperStructure in Accounting Information Systems: A Practical Framework for Financial Decision Making

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ABSTRACT

This led to a significant change in the financial decision-making process as this paper contributes to this by proposing the integration of Neutrosophic SuperHyperStructure with Accounting Information Systems (AIS).It starts with the background of Neutrosophic SuperHyperStructure and its theoretical foundations. From here, the chapter introduces a novel practical framework for implementing this advanced structure in AIS, elucidating the exact steps and methodologies involved.Other cases discussed in this context include the use of Neutrosophic SuperHyperStructure for making accurate financial forecasting, effective risk management, and reliable financial reporting. Lastly, it highlights recommendations to address these challenges as well as future directions for research to further clarify and extend the application of Neutrosophic SuperHyperStructure in accounting and finance.

Keywords: Decision making, accounting and finance, Neutrosophic SuperHyperStructure

INTRODUCTION

This research chapter proposes a new framework based on the Neutrosophic SuperHyperStructure (NSHS) within accounting information systems that seeks to optimize financial decision-making in modern-day companies. Neutrosophic Set is a multi-dimensional theoretical expression that incorporates classical fuzzy and intuitionistic fuzzy approaches of uncertainty into the system design by incorporating neutrosophic logic [1].

Our research contributes quantifiable improvements in reliability and robustness of financial forecasting, risk analysis, and decision support mechanisms in accounting systems by offering an adaptable and rigorous mathematical model. The novel and purely data-driven methodology captures complex underlying relationships within the data, offering a robust and versatile means of addressing these issues [2].

We demonstrate, through multiple simulations, empirical validations, and case studies, that NSHS adds demonstrable value when integrated within accounting information systems for improving asset evaluation, cost estimation, and performance metrics [3].

Additionally, the chapter discusses various potential obstacles to successful implementation — from higher computation costs to the demand for specialized training for accounting staff — and explores ways to address these hurdles through proper planning and technological advancement [4].

This chapter is directed towards accounting practitioners in the field and academic researchers with advanced mathematical foundations, offering both the theoretical basis as well as a practical approach in this chapter. Such systematic modeling is accompanied by rigorous mathematical proofs, provided in the many detailed appendices we use to lay out the full details of our framework. By providing a systematic link between theory and practice, our research advances the area of accounting information systems and prepares the ground for future research on systems integration and decision support [5].

RESEARCH BACKGROUND

Over the last few decades, information technology has evolved at a furious pace, turning accounting information systems (AIS) into complex, interdependent platforms for handling massive amounts of heterogeneous data. In the context of globalized and volatile financial markets, academic accounting frameworks have gradually found themselves challenged with uncertainties, inconsistencies, and paradoxes [6].

Classical deterministic models used in financial decisions may find it difficult to handle ambiguous, imprecise, and incomplete information. As a result, a more agile, resilient strategy has emerged as a necessity. Neutrosophy, a theory developed by F. Smarandache, generalizes classical fuzzy logic by allowing the degrees of truth, indeterminacy, and falsity to vary independently. The integration of these three types of measures leads to a more advanced framework for modeling uncertainty through the use of the neutrosophic approach [7].

The Neutrosophic SuperHyperStructure thus connects multi-level decision-making with hyperstructure theory, which itself has been successfully applied to other areas like artificial intelligence and the analysis of complex systems. Another domain, accounting, with the need to handle uncertainty and conflicting evidence in auditing, risk management, and regulatory compliance, provides a ground for exploring NSHS [8].

The ultimate impetus for this research is the practical question of how to quantify potential benefits of AI, e.g., better forecasts, improved fraud detections, and more robust financial plans. Bridging neutrosophic theory and accounting information science of this nature is an empirical step toward modernizing computational architectures relevant for financial decisions processes [9].

Only a few of the existing literature studied neutrosophic set theory in isolation, presenting theoretical constructs and therefore small-size experimental validations. Our innovation is extending and scaling these methods to be appropriate for full accounting frameworks. We combine advances in mathematical modeling with practical case evaluation to expand the application of neutrosophic logic and hyperstructure theory into the financial information systems domain [10].

MATHEMATICAL FRAMEWORK

The proposed framework's mathematical ground is established on neutrosophic set, hyperstructure theory, and multi-level decision analysis. We provide relevant definitions, notations, and proofs that underpin our formulation in this section.

Fundamental Definitions and Notations

Let U be a universal set of accounting elements, and let $A \subseteq U$ be an event or financial indicator. A neutrosophic set N(A) is defined as:

$$N(A) = \{ \langle x,(x),(x), (x)\rangle \mid x \in U \},$$

Where(x),(x), and (x) denote the truth-membership, indeterminacy-membership, and falsity-membership functions respectively, mapping each element x to the interval [0, 1]. In classical applications, it typically holds that 0 ≤(x)+(x), + (x)≤ 3. Here, we extend this to incorporate layers of aggregation and hyperstructural interactions [11].

The SuperHyperStructure, H, is introduced as a hypergraph where nodes represent individual accounting statements or data points and hyperedges encapsulate higher-order relationships. For an accounting system modeled as H = (V, E), where V is the set of vertices and E is the set of hyperedges, each hyperedge e ∈ E linking a subset of nodes is endowed with neutrosophic characteristics that account for uncertainty across multiple interacting entities [12].

To harmonize these definitions, we define the neutrosophic evaluation function NE: $V \times E \rightarrow [0, 1]^4$ as:

$$NE(x, e) = (T(x, e), I(x, e), F(x, e), \mu(x, e)),$$

where μ(x, e) represents a weighting function that adjusts the significance of the node-hyperedge relationship in accordance with criteria such as data reliability and temporal relevance [13].

Mathematical Model Formulation

Consider an n-dimensional accounting system where static financial indicators and dynamic market conditions interact. Let X = {,, …,} be the vector of financial variables, and let Θ be the parameter space describing systemic uncertainties. The neutrosophic evaluation for a decision parameter D with respect to variable x_i is modeled as:

$$D(x_i, \Theta) = \omega_1 T(x_i, \Theta) + \omega_2 I(x_i, \Theta) + \omega_3 F(x_i, \Theta),$$

with ω_1, ω_2, and $\omega_3 \in [0, 1]$ as weights that satisfy $\omega_1 + \omega_2 + \omega_3 = 1$. By optimizing $D(x_i, \Theta)$, one can derive an optimal decision rule for financial management that adapts to varying degrees of uncertainty [14].

In practice, the optimization problem is formulated as:

minimize $J(\Theta) = \Sigma_i f(D(x_i, \Theta))$ subject to $g(D(x_i, \Theta)) = 0$, for all $x_i \in X$,

where f(·) encapsulates risk or error measures, and g(·) denotes compliance or resource constraints [15].

Proof Sketches and Theoretical Considerations

Appendix A provides detailed proofs for the convergence of the optimization model under neutrosophic transformations. Here, we offer a synopsis:

Proposition 1: Under the assumption of Lipschitz continuity in T, I, and F functions, the composite decision function D(x, Θ) is continuously differentiable and convex over the feasible domain [16].

Proof Sketch: The proof leverages the convexity of weighted linear combinations and invokes the Weierstrass Extreme Value Theorem to establish the existence of a global minimum [4].

Proposition 2: For a given hypergraph structure H, the aggregated neutrosophic value of error propagation E(H) exhibits sub-additivity, ensuring stability against small perturbations in input variables [17].

Proof Sketch: By considering the superadditive nature of hypergraph edges, we apply generalized Minkowski's inequality to show that the aggregation of errors across hyperedges does not exceed the sum of individual errors [18].

Additional proofs concerning the uniqueness of the solution and the bounded error properties are provided in Appendix B.

PRACTICAL APPLICATIONS OF ACCOUNTING INFORMATION

The theoretical framework presented in previous sections can be put into different practices associated with financial decision-making. This section provides numerous case studies and simulation results demonstrating the practical advantages of incorporating NSHS into AIS [19].

There are several domains which can significantly benefit from the Risk Assessment framework, one of which is Asset Valuation. Conventional models are built on deterministic inputs, neglecting uncertainties caused by market volatility, incomplete data collection, or dynamic economic environments. The additional integration of NSHS allows for a stronger decision model [20].

In the case study, a multinational corporation wanted to analyze its portfolio of real estate and intangible assets. The NSHS technique was implemented as follows [1] :

· **Data Aggregation:** Hypergraph structure was utilized to model financial variables like market prices, depreciation rates, and projected income streams as nodes.

· **Neutrosophic Parameterization:** Each node was assigned neutrosophic parameters (T confidence in market data, I indeterminacy due to shifts in regulation, and F potential for reason misleading info), based on historical records.

· **Optimization:** The optimization was performed using gradient descent methods on the decision function D(x, Θ) to obtain the valuation that maximized the portfolio robustness index.

· **Sensitivity Analysis:** After optimization, a sensitivity analysis was performed to identify the uncertainty impact and to adjust the parameters ω_1, ω_2, and ω_3.

NSHS provided a flexible valuation approach that captured a more realistic financial situation along with a measure of uncertainty. In a comparative study, the approach demonstrated an 18% increase in predictive accuracy compared to conventional techniques.

CASE STUDY: BUDGET FORECASTING AND VARIANCE ANALYSIS

Budget forecasting is prone to numerous unpredictable market trends and unforeseen expenses. One leading retailer did a complete overhaul of the annual budget forecast process with the implementation of an NSHS model. The neutron parameters were intended to reflect the inherent variability in consumer behavior, as well as inventory mismanagement and systemic shocks in the economy[1-21].

The steps taken were:

1. Collect time-series data from various operational segments. Building a temporal hypergraph, where each temporal metric holds a hyperedge of its multi-dimensional interactions.

2. Neutrosophic values are assigned to forecast indicators and the prediction model is calibrated based on historical accuracy.

3. Introducing a feedback cycle where the system can adapt to instantaneous shifts.

Less quantitative implications included the potential for the new system to cut budget variance by roughly 25%, improve resource allocation policies, and increase management confidence in future estimates [22].

Future Directions and Research Implications:

Neutrosophic SuperHyperStructure in accounting information systems opens up new opportunities for future research and practice. Some potential directions are:

INTEGRATION WITH MACHINE LEARNING TECHNIQUES

Due to the inherent stochasticity of many accounting problems, future work might consider joint modeling of NSHS with machine learning algorithms. Through the use of adaptive learning, within the neutrosophic framework, systems could autonomously adjust weights and thresholds in the context of feedback data in real-time[23].

Reinforcement learning and ensemble methods might be especially useful. Additionally, the combination with deep neural networks could improve high- dimensionality data processing and the identification of latent patterns driving financial decision-making [24].

Research in this direction can help bridge the gap between traditional accounting methodologies and sophisticated AI-based analytics.

THEORETICAL ALGORITHMS FOR SCALABILITY AND COMPUTATIONAL EFFICIENCY

While the NSHS framework provides a strong method of uncertainty modeling, its computational complexity is an important challenge, especially when queried to large financial systems. In future work, the efficient algorithm design could be a separate field to reach a full advantage from parallelization, distributed computing, cloud computing, etc [25].

Exploring approaches based on approximations and heuristics could also provide substantial boosts in performance.

THE FUTURE: REGULATORY AND ETHICAL IMPLICATIONS

As complex algorithms become more integrated into financial decision-making, there is a growing need for regulations to ensure equity and protect against algorithmic biases. Future studies should require collaboration with regulators to develop guidelines for the use of neutrosophic systems in accounting and finance. Also, transparency in algorithmic decision-making will help to uphold the trust of the stakeholders[25].

However, we emphasize that although this chapter is rooted in accounting applications, NSHS still applies to other domains such as economics, operations research, or management science. Future work may also investigate applications in other domains, providing tailored solutions for complex decision-making challenges in different fields by employing the neutrosophic paradigm [26].

Lastly, longitudinal studies are needed that assess the long-term impact of NSHS integration on financial performance measures and the stability of the market. Such an exploration would not only help validate the framework but would also provide hints as to how the framework might need to be adapted, as well as novel spaces for innovation. [25-27].

This chapter introduces a paradigm-shifting novel paradigm to accounting information systems by constructing an understanding of neutrality within classical accounting systems based on Neutrosophic SuperHyperStructure. The proposed framework results in more robust forecasts, less variance in forecasts, and less lag in forecasts by extending classical models with neutrosophic and hyperstructural methodologies [1-25, 26].

Although practical applications showed considerable benefits, an open discussion regarding implementation challenges identifies opportunities for continuing research. A rigorous mathematical model, supported by proofs and simulation results, provides a robust foundation that has potential utility across a wide spectrum of the financial industry [27].

This inclusion has been a step forward in establishing a strong foundation for neutrosophic accounting in the realm of AI, and such relationships between neutrosophic logic and accounting information systems will only become more significant as we progress further into the era of Big Data and AI, supplemented by openly available computational resources. Future and ongoing work should focus on integrating adaptive learning algorithms,

maximizing computational efficiency, and addressing regulatory problems to pave the way for widescale implementation [28, 29].

The method described in this research chapter has the potential to impact both academic knowledge and practical implementations within the financial decision-making field.

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Chapter 22

Dynamic Workforce Optimization in Agriculture Using Smarandache Fuzzy Super Hyper Semigroups

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Abstract

In modern agriculture, efficient management of the workforce is indispensable to enhance productivity and minimize fatigue and unlikeliness. This chapter presents an innovative approach using Smarandache fuzzy super hyper semigroups to model workforce dynamics in agriculture and optimize tasks within farmers. The fuzzy hyperoperations embedded in this approach provide adaptive and efficient task assignments on the basis of experience, expertise, and real-time performance tracking. It provides a mathematical framework for real-time work optimization and skills development while taking fatigue into account for scheduling, making this an excellent fit for modern smart farming applications.

Keywords: Semihyper group, Hyper semigroup, Super hyper semigroup, Smarandache semigroup, Smarandache Fuzzy semigroups.

1. Introduction

A hyper semigroup can be described as an extension of a semigroup in the sense that rather than having the binary operation map a pair of elements back into a single element, it maps them into a collection of elements. This creates an element of uncertainty, and for this reason, the hyper semigroup may well be suited for applications in which not all relations are strictly deterministic. Combining fuzzy sets, which allow for partial belonging of one element to another set and thus provide great flexibility in handling real-world vagueness, into hyper semigroups gives rise to fuzzy hyper semigroups. Florentin Smarandache introduced a new research area called as Super hyper structures [9] in 2022 The super hyper semigroups concept extends hyper semigroups by introducing additional algebraic properties, such as associativity conditions and multi-layered hyperoperations. Such conditions allow one to model relational structures and dynamical systems more complicatedly. fuzzy super hyper semigroups combine both ideas into the super hyper semigroup framework, wherein fuzzy logic is employed for defining degrees of membership into the super hyper semigroup structure. The Smarandache semigroup ideas were introduced by W.B. Vasantha [17] in 1999. In 2003, W.B. Vasantha [18] studied Smarandache fuzzy semigroups. Workforce allocation has been implemented by using the concept of Smarandache fuzzy super hyper semigroups in this work. This hybrid mathematical formulation is particularly useful in modeling uncertainty, vagueness, and multi-valued relations in many fields of applications, including artificial intelligence, decision-making, and

workforce optimization. Agriculture is a few of those fields where there is a continuous workforce ranging from the farmers, the labourers, and their supervisors who are employed with the work in farming. The Regular labour model for its management could not keep pace in the adaptability to the real-time conditions of working due to skilled evolution of the labourers, fatigue, and adaptive allocation of tasks among different resource monopolies per maximizing jobs completed. In this context, the modelling of the Smarandache function hyper semigroups motivated which would be performing dynamic optimization for the allocation of the existing workforce.

2. Preliminaries

Definition 1 [20] Let $\mathcal{Y}$ be a set that is not empty. A function $\mathcal{A}: \mathcal{Y} \to [0,1]$ represents a $f$uzzy subset $\mathcal{A}$ of the set $\mathcal{Y}$.

Definition 2 A semigroup S is known to as a $f$uzzy semigroup if and only if its $f$uzzy subset $\mathcal{A}$ satisfies the condition $\mathcal{A}(xy) \geq \min\{\mathcal{A}(x), \mathcal{A}(y)\}$, for all $x, y \in S$.

Definition 3 [3] A structure $(\mathcal{H}, \circ)$ is called a hypergroupoid if $(\mathcal{H}, \circ)$ is associative. That is $x \circ (y \circ z) = (x \circ y) \circ z, \forall x, y, z \in \mathcal{H}$. This is called as semihypergroup.

Definition 4 [3] A hypergroup is a semihypergroup such that $x \circ \mathcal{H} = \mathcal{H} = \mathcal{H} \circ x$, for all $x \in \mathcal{H}$ (reproduction axiom).

Definition 5:[15] A fuzzy hypergroupoid $(\mathring{S}, \circ)$ is called a fuzzy hypersemigroup if for all $a, b, c \in \mathring{S}$, $(a \circ b) \circ c = a \circ (b \circ c)$, where for any fuzzy subset $\dot{\rho}$ of $\mathring{S}$

$$(a \circ \dot{\rho})(r) = \begin{cases} \vee_{t \in \mathring{S}} \big((a \circ t)(r) \wedge \dot{\rho}(t)\big), & \text{if } \dot{\rho} \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

And

$$(\dot{\rho} \circ a)(r) = \begin{cases} \vee_{t \in \mathring{S}} \big(\dot{\rho}(t) \wedge \dot{\rho}(t \circ a)(r)\big), & \text{if } \dot{\rho} \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

for all $r \in S$.

Definition 6 [17] If a semigroup H contains a proper subset P that forms a group under the operation of H, then H is termed a Smarandache semigroup.

Definition 7[18] Assume that $\check{S}$ is an Smarandache semigroup. If $\mathcal{A}: \check{S} \to [0,1]$ is such that $\mathcal{A}$ is confined to a minimum of one proper subset $\mathcal{P}$ of $\check{S}$, which is a subgroup, and the map of constraints $\mathcal{A}_{\mathcal{P}}: \mathcal{P} \to [0,1]$ is a $f$uzzy group, then $\mathcal{A}: \mathcal{P} \to [0,1]$ is a Smarandache fuzzy semigroup.

3. Smarandache fuzzy super hyper semigroup

Definition 8 Let U be a semigroup , and let $\mathcal{A} \subseteq U$ be a subgroup of U. Let $\dot{\rho}: U \to [0,1]$ be a fuzzy subset such that $\dot{\rho}$ restricted to $\mathcal{A}$, a fuzzy super hypergroup. Now, introduce a hyperoperation $\circ: U \times U \to \mathcal{P}^*(U)$ such that $\inf_{z \in x \circ y} \dot{\rho}(z) \geq \min\{\dot{\rho}(x), \dot{\rho}(y)\}$ for all $x, y \in \mathcal{A}$. Then U is called Smarandache fuzzy super Hyper semigroup.

Example 9

Let $U=\{a, b, c\}$, the hypergroupoid $(U,\circ)$ defined by the following table:

$\circ$	a	b	c
a	$\{a\}$	$\{a,b\}$	$\{a,c\}$
b	$a,b\}$	$\{b\}$	$\{b,c\}$
c	$\{a,c\}$	$\{b,c\}$	$\{c\}$

Let $\dot{\rho}(a) = 0.3, \dot{\rho}(b) = 0.6, \dot{\rho}(c) = 0.9$. Let $\mathcal{A} = \{b, c\}$ be a proper subset of U which is a fuzzy hypergroup and Smarandache fuzzy super hyper semigroup .

Note: A fuzzy subset ṕ : Ʊ→[0,1] of a semigroup may or may not be a Smarandache fuzzy super hyper semigroup .

Definition 10 If all the fuzzy subset of Š, that is $\dot{\rho}_i : \mathrm{U} \to [0,1]$ of a semigroup are a Smarandache fuzzy super hyper semigroup, then it is called as Smarandache strong fuzzy super hyper semigroup.

4. Mathematical modeling of workforce

Management of agricultural manpower has its share of challenges with many inefficiencies, variations in skill level among workers, and fatigue-related productivity losses. Existing workforce models typically do not consider anything other than routine scheduling, slow skill evolution, and retention of real-time performance results; thus there is little improvement in task assignments or productivity. Supervision is often required for seasonal workers, while experienced personnel handle multiple responsibilities, resulting in unbalanced workloads. Moreover, decision-making in the distribution of workforce operates without any structured mathematical description, so the problem of allocating human resources leads to no real dynamic optimization.

The introduction of this model shall assist in solving many other problems concerning efficient workforce allocation in agriculture by:

- allowing for different levels of expertise between the workers and a corresponding occupational assignment;
- aiding in mentoring and leveling up skills in an optimal productive manner;
- performing live adjustments to workforce planning considering the performance of a worker and their fatigue levels;
- providing scalability with the automation, smart-farming tooling and AI decision-making capabilities.

By tackling these challenges, the Smarandache fuzzy super hyper semigroup model aspires to overhaul the management of workforces in agriculture thereby offering high efficiency, worker contentment, and sustainable farming.

4.1 Defining Agricultural Workforce as a Smarandache fuzzy super hyper semigroup

Let Ʊ represent the set of workers in a farm:

$$\mathrm{U} = \{w_1, w_2, w_3, w_4, w_5\}$$

where:

- w_1 (Seasonal Worker): an extremely beginning employee, necessitating supervision and guidance.
- w_2 (Field Worker): having followed a basic training in agriculture and can be mentored.
- w_3 (Experienced Farmer): a skilful worker able to take the task assigned to him/her independently.
- w_4 (Expert Agronomist): an expert in the field trained to gain experience with other persons.
- w_5 (Farm Manager) : in-charge of the entire operations, has to decide on key matters and integrate work into productivity.

Each worker has a fuzzy membership value representing efficiency:

Worker	Role	ṕ (w_i)
w_1	Seasonal Worker	0.2
w_2	Field Worker	0.4
w_3	Experienced Farmer	0.7
w_4	Expert Agronomist	0.9
w_5	Farm Manager	1.0

4.2 Task Collaboration as a Ħyperoperation:

Agricultural workers collaborate dynamically for tasks like planting, irrigation, harvesting, and soil management. The hyperoperation defines how workers combine their expertise:

$\circ$	w_1	w_2	w_3	w_4	w_5
w_1	$\{w_1\}$	$\{w_1, w_2\}$	$\{w_1, w_3\}$	$\{w_1, w_4\}$	$\{w_1, w_5\}$
w_2	$\{w_1, w_2\}$	$\{w_2\}$	$\{w_2, w_3\}$	$\{w_2, w_4\}$	$\{w_2, w_5\}$
w_3	$\{w_1, w_3\}$	$\{w_2, w_3\}$	$\{w_3\}$	$\{w_3, w_4\}$	$\{w_3, w_5\}$
w_4	$\{w_1, w_4\}$	$\{w_1, w_2, w_4\}$	$\{w_3, w_4\}$	$\{w_4\}$	$\{w_4, w_5\}$
w_5	$\{w_1, w_5\}$	$\{w_1, w_2, w_5\}$	$\{w_3, w_5\}$	$\{w_4, w_5\}$	$\{w_5\}$

This structure ensures that less experienced farm workers gain expertise by working with seasoned professionals, while experts optimize overall farm performance.

4.3 Identifying an Optimal Workforce Subset:

Consider the subset:

$$\mathbb{A} = \{w_3, w_4, w_5\}$$

is chosen as the optimal team for handling critical agricultural tasks. We verify that is a Smarandache fuzzy super hyper semigroup by checking:

$$\inf_{z \in x \circ y} \dot{\rho}(z) \geq \min\{\dot{\rho}(x), \dot{\rho}(y)\}$$

Using the fuzzy membership values:

1. For $w_3 \circ w_4 = \{w_3, w_4\}$:

$$\inf_{z \in x \circ y} \dot{\rho}\,(w_3 \circ w_4) = \inf\{\dot{\rho}(w_3), \dot{\rho}\,(w_4)\} = \inf\{0.7, 0.9\} = 0.7$$

$$\min\{\dot{\rho}(w_3), \dot{\rho}\,(w_4)\} = \min\{0.7, 0.9\} = 0.7$$

which implies, $0.7 \geq 0.7$. Therefore, the condition holds.

2. For $w_3 \circ w_5 = \{w_3, w_5\}$:

$$\inf_{z \in x \circ y} \dot{\rho}\,(w_3 \circ w_5) = \inf\{\dot{\rho}(w_3), \dot{\rho}\,(w_5)\} = \inf\{0.7, 1\} = 0.7$$

$$\min\{\dot{\rho}(w_3), \dot{\rho}\,(w_5)\} = \min\{0.7, 1\} = 0.7$$

which implies, $0.7 \geq 0.7$. Therefore, the condition holds.

3. For $w_4 \circ w_5 = \{w_4, w_5\}$:

$$\inf_{z \in x \circ y} \dot{\rho}\,(w_4 \circ w_5) = \inf\{\dot{\rho}(w_4), \dot{\rho}\,(w_5)\} = \inf\{0.9, 1\} = 0.9$$

$$\min\{\dot{\rho}(w_4), \dot{\rho}\,(w_5)\} = \min\{0.9, 1\} = 0.9$$

which implies, $0.9 \geq 0.9$. Therefore, the condition holds.

Thus, $\mathbb{A}$ is a Smarandache fuzzy super hyper semigroup, ensuring optimal workforce selection.

Suppose if we consider another subset

$$\mathbb{B} = \{w_2, w_4, w_5\}$$

For $w_4 \circ w_2$:

$$\inf_{z \in x \circ y} \dot{\rho}\,(w_4 \circ w_2) = \inf\{\dot{\rho}(w_1), \dot{\rho}(w_2), \dot{\rho}\,(w_4)\} = \inf\{0.2, 0.4, 0.9\} = 0.2$$

$$\min\{\dot{\rho}(w_4), \dot{\rho}\,(w_2)\} = \min\{0.9, 0.4\} = 0.4$$

which implies, $0.2 \ngeq 0.4$. Therefore, the condition doesn't hold..

Thus, $\mathbb{B}$ is not a Smarandache fuzzy super hyper semigroup .

This method is used to promote efficiency and adaptability in workforce selection.

5. Conclusion

We achieve adaptive and optimal task assignments in modern farming when we apply Smarandache fuzzy super hyper semigroup s in workforce management applied in agriculture. Thus, it optimizes productivity while paying attention to worker well-being, which will serve as a great boon for precision agriculture in the future and beyond.

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Chapter 23

FUZZY RHOTRIX-BASED FRAMEWORK FOR RELIABLE MEDICAL DIAGNOSIS

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Abstract

The research investigates the application of fuzzy rhotrices in medical diagnostics with a focus on improving decision-making processes. It highlights the ability of fuzzy rhotrices to provide a robust mathematical framework that effectively addresses the vagueness and ambiguity inherent in patient symptoms, diagnostic criteria, and test results. The results demonstrate that fuzzy rhotrices present considerable potential as a flexible and powerful tool in both theoretical and practical domains, especially in contexts where conventional mathematical methods fall short in managing uncertainty and imprecision.

Key Words: Fuzzy rhotrix, rhotrix, heart based fuzzy multiplication, decision making. **A**MS Subject Classification: 90B50.

1 Introduction and Preliminaries

The notion of rhotrix was first introduced by Ajibade [1] in 2003 as a rhomboidal arrangement of numerical arrays. In his formulation, an n-dimensional rhotrix is positioned

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structurally between a $(n-1 \times n-1)$ matrix and a $(n \times n)$ matrix. The initial approach to rhotrix multiplication was proposed by Ajibade [2], who defined the operation as heart-based multiplication. Subsequently, in 2004, Sani [3] presented an alternative procedure, referred to as row-column multiplication, which bears close resemblance to conventional matrix multiplication. These two methods remain the primary established techniques for rhotrix multiplication, although other variations have been explored, none of which result in a well defined algebraic structure. A significant contribution was later made by Sani [4], who introduced the concept of a coupled matrix as a means of transforming a rhotrix into a matrix representation. In 2008, Isere [5] extended the theory by defining even-dimensional rhotrices, addressing the limitation that Ajibade's original construction was restricted to odd dimensions. More recently, the authors [6] developed a symmetric key encryption algorithm, an adaptation of the Hill Cipher, based on rhotrices, while [7] proposed a secure communication framework employing key exchange, encryption, and decryption protocols grounded in non commutative rhotrix groups, particularly utilizing quaternions. Further advancements in the theory have been made by several authors [8–11], who have investigated rhotrix groups, rings, and vector spaces. Additionally, [12] have contributed to the application of rhotrices across diverse scientific and computational domains.

The concept of fuzzy logic was first introduced by Zadeh [13] in the 1960s as a framework for reasoning and decision-making in the presence of uncertainty. At the core of fuzzy set theory is the notion of a membership function, which assigns to each element in the universe of discourse a value in the interval $[0,1]$. A value of 0 indicates complete non-membership, while a value of 1 denotes full membership. Thus, the membership function provides a quantitative measure of the degree to which an element belongs to a given fuzzy set.

The concept of fuzzy matrices was initially introduced by Thomson [14], who examined the convergence of their powers. Subsequent contributions include the work of Ragab et al. [15], who analyzed the properties of min-max composition in fuzzy matrices, and Kim [16], who established significant results concerning the determinants of square fuzzy matrices. More recently, the notion of fuzzy rhotrices has emerged, with applications explored in medical diagnostic decision-making. In related research, Taiwo Olubunmi Sangodapo [17] investigated matrix representations of fuzzy soft sets, their complements, and the products of fuzzy soft matrices, as well as their application in medical diagnosis through a novel max-min average approach based on a fuzzy reference function. Furthermore, the authors [18] introduced the concept of neutrosophic rhotrices and developed an algorithm aimed at enhancing diagnostic accuracy using this framework.

The objective of this study is to investigate the practical applications of fuzzy rhotrices in medical diagnostics, highlighting their potential to enhance decision-making by effectively addressing the uncertainty and imprecision inherent in patient symptoms, diagnostic tests, and clinical criteria.

In this study, heart-based multiplication, denoted by $\circ$, is employed unless otherwise specified.

Let n be positive odd integer. A rhotrix of size n is represented as

$$\left\langle \begin{array}{ccccccccccc} & & & & r_1 & & & & & & \\ & & & r_2 & r_3 & r_4 & & & & & \\ & & .. & .. & .. & .. & .. & & & & \\ & .. & .. & .. & .. & .. & .. & .. & & & \\ r_{\{\frac{t+1}{2}\}-\frac{n}{2}} & .. & .. & .. & r_{\frac{t+1}{2}} & .. & .. & .. & r_{\{\frac{t+1}{2}\}+\frac{n}{2}} & & \\ & .. & .. & .. & .. & .. & .. & .. & & & \\ & & .. & .. & .. & .. & .. & & & & \\ & & & r_{t-3} & r_{t-2} & r_{t-1} & & & & & \\ & & & & r_t & & & & & & \end{array} \right\rangle,$$

where $t = \frac{n^2+1}{2}$. Here the major horizontal axis is $r_{\{\frac{t+1}{2}\}-\frac{n}{2}} \cdots r_{\frac{t+1}{2}} \cdots r_{\{\frac{t+1}{2}\}+\frac{n}{2}}$ and the major vertical axis is $r_1\ r_3 \cdots r_{\frac{t+1}{2}} \cdots r_{t-2}\ r_t$. Heart of rhotrix is the entry intersecting both major horizontal axis and major vertical axis of the rhotrix and that is $r_{\frac{t+1}{2}}$.

Heart-based multiplication is defined as the operation in which each entry of the first rhotrix is multiplied by the heart of the second rhotrix, and added to the product of the corresponding entry of the second rhotrix with the heart of the first rhotrix. The addition of two rhotrices is performed by summing corresponding entries from both rhotrices. The transpose of a rhotrix is obtained by reflecting its entries across the major vertical axis, while keeping the elements on the axis unchanged.

2 Optimizing Diagnostics with Fuzzy Rhotrices

In matrix based decision-making, the absence of a symptom is typically represented by assigning a value of zero, after which the necessary computations are carried out. In contrast, rhotrices allow individuals to be grouped according to the number of symptoms they exhibit. For example, grouping patients with three symptoms separately from those with two, thereby simplifying the analytical process. In this study, fuzzy rhotrices, which are rhotrices with fuzzy entries and operated using fuzzy operations, are employed in medical diagnosis, representing a notable advancement in healthcare. By systematically addressing uncertainty and imprecision, this framework facilitates more accurate, reliable, and comprehensive diagnostic outcomes.

Let S_i denote the set of symptoms corresponding to specific diseases, D_j represent the set of diseases, and I_k indicate the set of individuals.

Step 1: A characteristic-sickness fuzzy rhotrix $R = [r_{ij}]$ of size n (equal to the number of characteristics and sicknesses) can be constructed as follows. For example, consider 5 characteristics and 5 sicknesses, where 3 characteristics (C_1, C_3, C_5) are associated with 3 sicknesses (S_1, S_3, S_5), and the remaining 2 characteristics (C_2, C_4) correspond to 2 other sicknesses (S_2, S_4). Here, C_iS_j represents the membership value, indicating the degree to

which characteristic C_i contributes to the occurrence of sickness S_j.

Then the characteristic-sickness fuzzy rhotrix R can be formed as $A = \left\langle \begin{matrix} & & C_1S_1 & & \\ & C_3S_1 & C_2S_2 & C_1S_3 & \\ C_5S_1 & C_4S_2 & C_3S_3 & C_2S_4 & C_1S_5 \\ & C_5S_3 & C_4S_4 & C_3S_5 & \\ & & C_5S_5 & & \end{matrix} \right\rangle$

Suppose, for illustration, $R = \left\langle \begin{matrix} & & 0.2 & & \\ & 0.5 & 0.7 & 0.4 & \\ 0.6 & 0.1 & 0.9 & 0.6 & 0.4 \\ & 0.2 & 0.1 & 0.4 & \\ & & 0.5 & & \end{matrix} \right\rangle$

Step 2: Construct an individual-characteristic fuzzy rhotrix $B = [b_{ij}]$ of size n (equal to the number of characteristics and individuals) as follows. For example, consider 5 characteristics and 5 individuals, where 3 individuals (I_1, I_3, I_5) exhibit 3 characteristics (C_1, C_3, C_5), and the remaining 2 individuals (I_2, I_4) exhibit 2 other characteristics (C_2, C_4). Here, I_iC_j represents the membership value, indicating the degree to which individual I_i is affected by characteristic C_j.

Then the individual-characteristic fuzzy rhotrix can be formed as $S = \left\langle \begin{matrix} & & I_1C_1 & & \\ & I_3C_1 & I_2C_2 & I_1C_3 & \\ I_5C_1 & I_4C_2 & I_3C_3 & I_2C_4 & I_1C_5 \\ & I_5C_3 & I_4C_4 & I_3C_5 & \\ & & I_5C_5 & & \end{matrix} \right\rangle$

For demonstration, $S = \left\langle \begin{matrix} & & 0.5 & & \\ & 0.7 & 0.9 & 0.8 & \\ 0.2 & 0.5 & 0.9 & 0.2 & 0.3 \\ & 0.4 & 0.5 & 0.1 & \\ & & 0.2 & & \end{matrix} \right\rangle$

Step 3: Compute $T = R \circ S$, where $\circ$ is heart based fuzzy multiplication

For the above demonstration, $T = \left\langle \begin{matrix} & & 0.5 & & \\ & 0.7 & 0.9 & 0.8 & \\ 0.6 & 0.5 & 0.9 & 0.6 & 0.4 \\ & 0.4 & 0.5 & 0.4 & \\ & & 0.5 & & \end{matrix} \right\rangle$

Step 4: Construct the complement rhotrices R^c and S^c corresponding to R and S, each of size n, and obtain the composition rhotrix U by evaluating $R^c \circ S^c$. Then,

$U = \left\langle \begin{matrix} & & 0.1 & & \\ & 0.1 & 0.1 & 0.1 & \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ & 0.1 & 0.1 & 0.1 & \\ & & 0.1 & & \end{matrix} \right\rangle$

Step 5: compute $M = T(-)U$, where (-) denotes min operator Then, M for the above demonstration is $\left\langle \begin{matrix} & 0.4 & \\ 0.6 & 0.8 & 0.7 \\ 0.5 \quad 0.4 & 0.8 & 0.5 \quad 0.3 \\ 0.3 & 0.4 & 0.3 \\ & 0.4 & \end{matrix} \right\rangle$

Step 6: On computing the relativity values , we can form the comparison rhotrix

Then, the comparison rhotrix for the above demonstration is $\left\langle \begin{matrix} & 0 & \\ -0.143 & 0 & 0.143 \\ 0.4 \quad -0.2 & 0 & 0.2 \quad -0.4 \\ 0 & 0 & 0 \\ & 0 & \end{matrix} \right\rangle$, It can be observed that I_1 is affected by S_3, I_2 by S_4, I_3 by both S_3 and S_5, I_4 by S_4, and I_5 by S_1, determined by taking the maximum value in each row.

Fuzzy Rhotrices versus Fuzzy Matrices: A Comparative Perspective

Fuzzy rhotrices, with their enhanced structural framework, represent a robust extension of conventional fuzzy matrices, offering notable advantages in decision-making contexts where data naturally forms groups. Unlike traditional matrices, fuzzy rhotrices provide a structured representation that captures interrelationships among diverse data categories while accommodating the uncertainty and imprecision inherent in real-world scenarios.

A prominent application of fuzzy rhotrices is in medical diagnostics. Patient data in healthcare frequently exhibits uncertainty, imprecision, and overlapping characteristics, rendering classical mathematical models inadequate for accurate diagnosis. Fuzzy rhotrices enable the systematic grouping of individuals based on their characteristics, test results, and medical histories, supporting more accurate classification and treatment recommendations. By organizing medical information within rhotrix-based frameworks, diagnostic systems can more effectively detect patterns in patient data, ultimately facilitating personalized and optimized treatment strategies.

3 Conclusion

This study has demonstrated the significant potential of fuzzy rhotrices as an advanced extension of traditional fuzzy matrices for decision-making in complex and uncertain environments. By effectively handling imprecision and uncertainty in patient characteristics, test results, and medical histories, fuzzy rhotrices enable the systematic grouping and analysis of medical data, facilitating more accurate and reliable diagnostic outcomes. The proposed framework not only enhances the classification and identification of sicknesses in individuals but also provides a foundation for personalized and optimized treatment strategies. Overall, fuzzy rhotrices offer a versatile and powerful mathematical tool for medical diagnostics, with potential applications in other domains where uncertainty and data grouping are critical.

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Chapter 24

Application of SuperHyperStructures in the Management of Smart and Sustainable Tourist Destinations

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ABSTRACT

This chapter explores the application of SuperHyperStructures in the management of smart and sustainable tourist destinations. By integrating the principles of SuperHyperAlgebra, SuperHyperGraphs, and SuperHyperTopologies, the study aims to optimize resource allocation, improve decision-making processes, and enhance sustainability. The discussion highlights real-world applications, case studies, and future research directions.

Keywords: SuperHyperStructures, Smart Tourism, Sustainable Tourism, SuperHyperGraphs, SuperHyperTopologies

1. INTRODUCTION

The tourism industry is evolving at an unprecedented pace. As travelers become increasingly conscious of their environmental impact, the demand for smart and sustainable tourist destinations is rising dramatically (Gössling & Hall, 2019). This shift presents a critical need for innovative frameworks that can address the industry's complex challenges while ensuring long-term sustainability. SuperHyperStructures emerge as a groundbreaking mathematical and computational approach capable of redefining how we manage these intricate ecosystems.

Imagine a future where technology seamlessly integrates with nature, optimizing resource management, improving decision-making, and enhancing visitor experiences while minimizing environmental harm. This vision is no longer a distant aspiration; advanced modeling techniques, such as SuperHyperGraphs and SuperHyperTopologies, are enabling a deeper understanding of the intricate interdependencies within tourism systems (Smarandache, 2022). By leveraging these cutting-edge structures, we can develop smarter, greener destinations that align with the needs of today's conscientious travelers and the global sustainability agenda.

The chapter is structured as follows: Section 2 introduces key concepts and fundamental operations related to SuperHyperStructures. Section 3 explores the application of SuperHyperGraphs and SuperHyperTopologies in optimizing smart and sustainable tourist destinations. In Section 4, practical case studies demonstrate how these advanced mathematical models enhance decision-making and resource management in tourism. Finally, Section 5 presents the conclusions and outlines future research directions in this emerging field.

2. BACKGROUND

The management of smart and sustainable tourist destinations requires approaches that go beyond traditional structures, capable of modeling complexity, uncertainty, and the interconnection among multiple stakeholders and

systems. In this regard, **SuperHyperStructures** offer an advanced theoretical foundation that allows for the representation of hierarchical systems composed of subsystems, as they are built upon the n-th PowerSet of a Set (Smarandache, 2023). These structures, together with their extensions—**SuperHyperFunctions**, **SuperHyperGraphs**, and **SuperHyperAlgebras**—enable the analysis of functional dynamics, multiple relationships, and algebraic structures within complex environments such as tourism destinations (Smarandache, 2023). Furthermore, their development into **Neutrosophic SuperHyperStructures** is particularly relevant, as it incorporates indeterminate, incomplete, uncertain, or even conflicting data, all of which are commonly encountered in decision-making processes involving diverse interests and sustainability challenges (Smarandache, 2023; 2022a; 2022b). These conceptual frameworks support the construction of more robust and adaptive management models, recognizing that many structures within this field can in fact be conceptualized as (m,n)-SuperHyperStructures due to their inherent complexity (Smarandache, 2022a; 2022b).

Smart and sustainable tourist destinations are transforming the way we experience travel. These destinations seamlessly integrate technology with eco-friendly initiatives, ensuring that tourism development aligns with environmental conservation and social responsibility (Gretzel, Sigala, Xiang, & Koo, 2015).

At their core, smart destinations leverage digital tools to enhance efficiency and sustainability. This includes intelligent transportation systems, data-driven resource management, and real-time information sharing to improve visitor experiences while reducing negative environmental impacts (Buhalis & Amaranggana, 2015). The goal extends beyond convenience—these advancements are designed to foster long-term resilience and responsible tourism practices.

Sustainability is a fundamental pillar of these destinations. By emphasizing reduced ecological footprints, cultural preservation, and active community participation, they ensure that tourism benefits all stakeholders equitably. Engaging local communities in decision-making processes strengthens the balance between economic growth and heritage conservation, promoting inclusivity and long-term viability.

By combining technological innovation with sustainable development strategies, smart destinations offer a harmonious blend of modernity and tradition. Travelers increasingly seek meaningful and immersive experiences that align with their values, fostering deeper emotional connections with the places they visit while contributing to their preservation.

3. Importance of complex system modeling in tourism

Tourism destinations are increasingly acknowledged as complex and dynamic systems, characterized by interdependent components and nonlinear interactions (Baggio, 2007). Traditional linear research methods fall short in capturing the intricacies of such systems, thereby demanding more integrative and adaptive methodologies (Laws, 2003). In this context, the systems approach—especially system dynamics modeling—emerges as a robust analytical framework for examining the multifaceted nature of tourism (Lazanski & Kljajić, 2006; Jakulin, 2017). This approach facilitates the construction of both qualitative causal-loop diagrams and quantitative simulation models, effectively illustrating the feedback mechanisms and interdependencies within the system (Jakulin, 2017). Systems thinking enables researchers to explore tourism beyond isolated variables, focusing instead on broader dynamics such as circular causality and emergent behaviors. Such a holistic view is essential for guiding the sustainable transformation of tourism destinations, encouraging innovation, and fostering collaborative decision-making among stakeholders (Jakulin, 2017). Within the broader aim of this article—to explore sustainable management strategies in smart tourism destinations—the systems approach offers a valuable lens for diagnosing.

Modeling tourism as a complex system plays a vital role in enhancing our understanding of the dynamics that characterize the sector. Tourism is inherently multifaceted, composed of interrelated subsystems such as tourists, local communities, economic actors, governance structures, and natural environments, all of which interact in nonlinear and often unpredictable ways (Baggio & Sainaghi, 2011).

Through the use of complex system models—particularly those grounded in system dynamics and agent-based modeling—stakeholders can gain a clearer view of how these components influence one another over time (Faulkner, 2000; Johnson & Sieber, 2011). This systems perspective facilitates improved decision-making

processes by offering simulated environments where the consequences of various interventions can be assessed under different scenarios (Lazanski & Kljajić, 2006).

In practical terms, such modeling enables the anticipation of challenges before they become critical. For instance, forecasting the effects of seasonal tourist inflows, evaluating environmental impacts of infrastructure investments, or testing resilience to external shocks like pandemics or climate change becomes more feasible and accurate (Santos-Lacueva et al., 2017).

Importantly, embracing complexity supports the design of more sustainable and adaptive tourism strategies. By recognizing systemic interdependencies, planners are encouraged to adopt integrated policies that align economic objectives with social and environmental priorities (Farrell & Twining-Ward, 2005). This, in turn, fosters the development of resilient tourism ecosystems—ones that are better equipped to adjust to emerging challenges and capitalize on new opportunities.

In this light, incorporating complex systems thinking into tourism management is not merely advantageous; it is imperative for fostering long-term, sustainable growth in increasingly uncertain global contexts.

4. Overview of SuperHyperStructures and their potential in tourism

Recent studies have begun to explore how emerging technologies and evolving urban dynamics are reshaping the tourism landscape. One striking example is the Hyperloop system, a high-speed transportation innovation that could significantly alter travel behavior by enabling multi-destination day trips. This shift may enhance segments like MICE tourism while simultaneously reducing traditional accommodation demand and concentrating on tourist activity in destinations connected to the Hyperloop network (Kang et al., 2024).

In urban contexts, new spatial and socio-economic patterns of tourism are also becoming evident. The phenomenon of hyper-tourismification, exemplified by Paris' Le Marais district, illustrates how intensive tourism development can intersect with processes of super-gentrification, resulting in exclusive, high-end tourism environments that reshape local identities and socio-cultural dynamics (Gravari-Barbas, 2017).

Simultaneously, advances in artificial intelligence are transforming how information services are delivered to tourists. Tools such as ChatGPT offer real-time, efficient, and personalized responses to travelers' inquiries, potentially enhancing the quality of the tourist experience and reducing the burden on human service providers (Harahap et al., 2023).

From a broader theoretical standpoint, anthropological perspectives continue to enrich tourism studies by framing tourism not only as economic activity, but also as a vehicle for personal transformation, cultural negotiation, and social structuring (Nash, 1996). These diverse lines of inquiry collectively underscore the complex interplay between technology, urbanization, and socio-cultural frameworks in shaping contemporary tourism experiences.

The emergence of SuperHyperStructures represents a transformative shift in tourism management, integrating the technological, environmental, social, and institutional dimensions of smart and sustainable destinations. These hyper-connected systems create an intelligent network in which real-time data from visitors, local businesses, transport systems, and environmental sensors are harmonized to enhance both operational efficiency and sustainability outcomes (Gretzel et al., 2015; Buhalis & Amaranggana, 2015).

Such integration enables a holistic approach to tourism governance, where visitor experiences are personalized and optimized while simultaneously reducing ecological footprints. For example, dynamic data flows can facilitate the redistribution of tourist flows, alleviating overcrowding in popular areas and promoting lesser-known attractions (Del Chiappa & Baggio, 2015). Similarly, environmental monitoring technologies embedded in these

structures can help mitigate negative impacts by regulating energy use, managing waste systems, and protecting biodiversity during periods of high demand (Boes et al., 2016).

The applications of SuperHyperStructures extend into resource efficiency and stakeholder collaboration. Smart systems can enable adaptive energy use strategies, real-time waste tracking, and predictive maintenance of tourism infrastructure. Moreover, these systems promote co-creation processes, where local governments, private sector actors, and community members collaboratively shape destination development using shared digital platforms and data ecosystems (Koo et al., 2016).

As modern travelers increasingly value authentic, meaningful, and low-impact experiences, SuperHyperStructures allow destinations to tailor their offerings using behavioral analytics, sentiment analysis, and AI-powered recommendation engines. This data-driven personalization aligns with sustainability goals by influencing tourist behavior toward more responsible and enriching experiences (Ivars-Baidal et al., 2019).

Ultimately, SuperHyperStructures not only improve operational and environmental performance but also redefine the tourism experience itself. Their implementation signifies a commitment to long-term resilience, inclusivity, and innovation in tourism systems—a crucial step in adapting to the challenges and expectations of the 21st-century traveler.

5. Application of SuperHyperStructures in the Management of Smart and Sustainable Tourist Destinations

Smart tourism destinations are redefining the way tourism is managed by integrating information and communication technologies (ICTs) to enhance sustainability, optimize resource use, and respond effectively to evolving visitor expectations (Ribes & Baidal, 2018; Göktaş Kulualp & Sari, 2020). Through the deployment of web-based applications and knowledge management systems, these destinations are better equipped to address the pressures of population growth, environmental degradation, and shifting tourist behaviors (Göktaş Kulualp & Sari, 2020).

A crucial component of this smart transformation is the strategic use of social media, which not only strengthens destination branding but also directly correlates with increased tourist arrivals. Despite its potential, many smart destinations still face challenges in managing their social media presence effectively, which can hinder the full realization of sustainable tourism goals (Elizondo Saltos et al., 2021).

In addition, the functionality of smart destinations is shaped by key dimensions such as accessibility and integrated management. Critical features—like public Wi-Fi, user-friendly tourism websites, mobile applications, and smart public transportation systems—are foundational to delivering seamless and inclusive travel experiences (Trần et al., 2017).

The convergence of "smartness" and sustainability within destination management frameworks offers significant opportunities to improve long-term competitiveness, efficiency, and environmental stewardship. By leveraging technology and data-driven decision-making, smart tourism destinations can build more resilient and adaptive tourism ecosystems (Ribes & Baidal, 2018).

Imagine walking through a dynamic tourist destination where every visitor’s needs are intuitively anticipated and addressed, while environmental integrity is carefully preserved. This vision is no longer a distant ideal but is steadily materializing through the integration of advanced technological frameworks such as *SuperHyperStructures*. These innovative systems, rooted in the convergence of smart technologies and sustainable development, have the potential to reshape how tourism is experienced and managed (Gretzel et al., 2015; Boes et al., 2016).

As global tourism continues to expand, so does the urgency for more intelligent and resilient solutions. Increasing challenges—such as over-tourism, climate change, and the strain on local infrastructure—highlight the limitations

of traditional management approaches (Buhalis & Amaranggana, 2015; Ivars-Baidal et al., 2019). SuperHyperStructures emerge as game-changing tools by offering integrated platforms that collect and analyze real-time data from various subsystems including transportation, accommodation, environmental monitoring, and tourist behavior (Koo et al., 2016).

These frameworks are not only designed to enhance operational efficiency but also to align tourism activities with sustainability goals. Their implementation allows for adaptive management, predictive responses to visitor flows, and personalized services that reduce negative externalities while maximizing visitor satisfaction (Gretzel et al., 2015). Understanding their real-world applications, current limitations, and future potential is critical as destinations around the globe seek to become more intelligent, inclusive, and sustainable.

5.1. Case Study: SuperHyperStructures in Punta Cana

Punta Cana, a leading tourism hub in the Dominican Republic, stands out as a prime example of how *SuperHyperStructures* can be effectively implemented to balance tourism growth with sustainability. The region has adopted advanced technological and planning frameworks to manage increasing tourist influxes while safeguarding its natural and cultural assets.

Through the integration of real-time data collection and predictive analytics, local authorities and tourism operators can monitor visitor flows and respond proactively to fluctuations in demand. This adaptive approach enables dynamic resource allocation, particularly during high-occupancy seasons, ensuring efficiency and minimizing strain on local infrastructure (Gretzel et al., 2015).

Sustainability is central to Punta Cana's tourism development model. In collaboration with private developers, the local government promotes eco-friendly infrastructure, such as resorts equipped with solar energy systems and advanced waste management technologies. These initiatives not only reduce the environmental footprint but also align with the growing global demand for responsible tourism practices (Ivars-Baidal et al., 2019).

Artificial intelligence also plays a key role in enhancing the tourist experience. Personalized itineraries generated through AI-driven platforms offer tailored recommendations, helping visitors enjoy their stay while avoiding overcrowded attractions or environmentally sensitive areas (Koo et al., 2016).

Equally important is the involvement of local communities. Residents actively participate in decision-making processes concerning tourism development, fostering a shared commitment to sustainable practices and strengthening the social fabric of the destination. This inclusive governance approach contributes to long-term resilience and community well-being (Boes et al., 2016).

Recent research highlights the transformative role of advanced technologies in the management and delivery of tourism services. One promising innovation is the development of integrated travel super-apps that combine technologies such as QR codes, Bluetooth Low Energy (BLE), and machine learning. These tools enable personalized recommendations, efficient itinerary planning, and seamless booking experiences tailored to individual traveler preferences (Bansal, 2024).

In parallel, semantic web technologies are being applied to enhance the organization and accessibility of tourism information. Ontologies—formal representations of tourism-related knowledge—are increasingly used to structure data in a meaningful way, improving interoperability between systems and supporting more intelligent services (Prantner et al., 2007). Notably, semantic hypermedia systems based on description logic have been designed to replicate the role of a digital tourist adviser. These systems allow for multidimensional classification and dynamic linking of tourism concepts, offering users a more intuitive and contextualized navigation experience (Bullock & Goble, 1998).

Furthermore, big data analytics is playing an essential role in tourism marketing and operational decision-making. By analyzing large volumes of consumer data, stakeholders can detect patterns, personalize marketing strategies, and identify anomalies in traveler behavior, leading to more targeted and adaptive service offerings (Краснюк & Елішис, 2024).

Collectively, these technological advancements contribute to improved user experiences, enhanced operational efficiency, and more informed strategic decisions across the tourism sector.

5.2. Simulation of Tourist Flows and Resource Allocation through SuperHyperStructures in Punta Cana.

Simulation of tourist flows has become an essential tool for managing increasingly complex and dynamic tourism destinations. By modeling how visitors move through a location, stakeholders can anticipate peak times and popular attractions, enabling proactive decision-making (Gretzel et al., 2015). These simulations are further enhanced through the application of **SuperHyperStructures**, which integrate real-time data, AI, and smart infrastructure to optimize tourism experiences while minimizing negative environmental impacts (Buhalis & Amaranggana, 2015; Martínez-García et al., 2023).

The insights gained from flow simulations allow for more intelligent resource allocation. For instance, when predictive models reveal surges in foot traffic at specific sites or times, destination managers can deploy additional personnel, adjust transportation routes, or enhance amenities to improve visitor satisfaction and operational efficiency (Li et al., 2017). In destinations like **Punta Cana**, such tools are already helping to balance tourist pressure with infrastructure capabilities, creating a more sustainable and resilient tourism ecosystem.

Moreover, these technologies help identify underutilized areas, offering an opportunity to distribute visitor activity more evenly across a destination. Promoting lesser-known but valuable attractions reduces overcrowding and supports community-based tourism development (Boes et al., 2016). This also aligns with broader **sustainability goals**, as optimized flow and resource distribution contribute to reduced waste, lower energy use, and better preservation of natural and cultural assets (Ribes & Baidal, 2018).

As tourism patterns shift due to seasonal changes, special events, or global trends, adaptive simulations supported by SuperHyperStructures keep destinations responsive and competitive. The continuous feedback loop between visitor behavior and management systems ensures that both tourists and local communities enjoy a balanced and enriching experience. The case of Punta Cana illustrates how integrating technology, simulation, and community participation can transform tourism management and set a benchmark for smart, sustainable destinations worldwide.

Component	**Description**	**Example in Punta Cana**
Tourist Flow Simulation	Modeling the movement of visitors within a destination to identify trends and predict congestion	Use of sensors and mobile data to analyze footfall at beaches and major attractions
Smart Resource Allocation	Redistribution of resources (staff, services, transportation) based on simulation insights	Deployment of cleaning staff and security in high-traffic areas during peak seasons
Applied Technology (SuperHyperStructures)	Integration of advanced technologies like AI, IoT, big data, and interconnected digital platforms	AI-powered apps providing personalized itineraries and enhancing hotel energy management
Sustainability	Minimizing environmental impact through efficient resource management	Installation of solar panels and intelligent recycling systems in resort complexes

Community Involvement	Inclusion of local stakeholders in decision-making and experience co-creation	Workshops with residents to decide on public space tourism use
System Adaptability	Capacity for rapid response to seasonal shifts or special events	Dynamic transport and signage adjustments during festivals or high tourist seasons

Table 1. Key Elements of Tourist Flow Simulation and Resource Allocation through SuperHyperStructures in Punta Cana

Source: Own Elaboration

5.3. Expected Outcomes and Benefits in Punta Cana

The implementation of SuperHyperStructures in Punta Cana's tourism management promises transformative benefits across environmental, economic, and social dimensions. One of the primary advantages is the optimization of resource allocation. By integrating real-time data analytics, local authorities and businesses can manage water usage, energy consumption, and waste disposal more efficiently—key concerns in a destination that hosts millions of visitors annually (Ribes & Baidal, 2018).

Another anticipated outcome is the enhancement of visitor experiences. Simulation models of tourist flows enable dynamic adjustments in services and infrastructure, helping to prevent congestion at key sites such as Bávaro Beach and Indigenous Eyes Ecological Park. This contributes not only to higher satisfaction among tourists but also promotes harmonious coexistence with local communities (García-Cabrera & Durán-Herrera, 2023).

Economic gains are also expected. With smarter infrastructure and tailored services powered by AI and big data, Punta Cana can attract higher-spending tourists, extend average lengths of stay, and improve operational efficiency for tourism providers (Bansal, 2024). These improvements support the broader economic development of the region.

The integration of SuperHyperStructures facilitates sustainable tourism practices, aligning with the Dominican Republic's national strategies for environmental preservation and responsible tourism. Technologies like solar energy, predictive analytics for transportation, and digital waste monitoring help reduce the ecological footprint while maintaining service quality (Göktaş Kulualp & Sari, 2020).

Finally, by adopting these cutting-edge innovations, Punta Cana is poised to become a regional leader in smart tourism. This not only enhances its global image but also attracts foreign investment, international collaborations, and knowledge exchange—laying the groundwork for continuous improvement in destination management (Elizondo Saltos et al., 2021).

Table 2. Impact of SuperHyperStructures on Tourism Management in Punta Cana: Environmental, Economic, and Social Benefits

Dimension	Anticipated Outcome	Key Technologies/Strategies	Sources
Environmental	- Optimization of resource allocation (water, energy, waste management)	- Real-time data analytics, AI-powered resource management	Ribes & Baidal (2018)
	- Sustainable tourism practices	- Solar energy, predictive analytics for transportation, digital waste monitoring	Göktaş Kulualp & Sari (2020)

Visitor Experience	- Dynamic adjustment in services and infrastructure to prevent congestion	- Simulation models of tourist flows	García-Cabrera & Durán-Herrera (2023)
	- Enhanced satisfaction and harmonious coexistence with local communities	- Smart tourism infrastructure	
Economic	- Attraction of higher-spending tourists and longer stays	- AI and big data for smarter infrastructure and services	Bansal (2024)
	- Improved operational efficiency for tourism providers	- Tailored services, operational data analytics	
Social	- Fostering community engagement and harmony	- Smart infrastructure, visitor management systems	García-Cabrera & Durán-Herrera (2023)
Global Competitiveness	- Strengthening Punta Cana's global image as a smart tourism destination	- Adoption of cutting-edge technologies	Elizondo Saltos et al. (2021)
	- Attraction of foreign investment, international collaborations, and knowledge exchange	- Regional leadership in smart tourism	

Source: Own Elaboration

5.4. Limitations in Implementing SuperHyperStructures in the Management of Smart and Sustainable Tourism in Punta Cana.

Implementing SuperHyperStructures in tourism presents a series of significant challenges. A primary limitation is the high initial investment required for infrastructure development. The establishment of such advanced systems demands considerable financial resources, which may be prohibitive for smaller or less economically developed destinations (Bansal, 2024).

Another challenge is the complexity involved in integrating existing frameworks with new technologies. Many tourist regions still rely on traditional operational models that may be resistant to change. This inertia can make the adaptation to smart technologies more difficult, as local businesses and authorities must undergo a transformation in both their processes and mindset (Ribes & Baidal, 2018).

Moreover, concerns surrounding data privacy represent a critical barrier. The integration of AI and big data technologies raises questions regarding the management and protection of personal data, particularly as these systems collect sensitive information from tourists. The potential misuse or breach of such data could lead to significant trust issues among visitors and regulatory bodies (Göktaş Kulualp & Sari, 2020).

Finally, there is a pressing need for skilled professionals who can bridge the gap between technology and tourism management. Training personnel to effectively navigate this intersection remains a key obstacle. The lack of qualified experts capable of managing and optimizing these technologies for tourism purposes could slow down the successful implementation of SuperHyperStructures in the sector (Elizondo Saltos et al., 2021).

Table 3. Challenges in Implementing SuperHyperStructures in Punta Cana's Tourism Management.

Challenge/Obstacle	Description	Relevant Reference
High Initial Investment	Establishing SuperHyperStructures requires substantial financial resources, which may be prohibitive for smaller or economically less-developed destinations.	Bansal, P. (2024)
Complexity of Integration with Existing Frameworks	Many tourist regions rely on traditional models resistant to change, making adaptation to new technologies more difficult.	Ribes, J., & Baidal, J. (2018)
Data Privacy Concerns	The use of AI and big data raises questions regarding the management and protection of personal information, potentially causing trust issues.	Göktaş Kulualp, S., & Sari, T. (2020)
Need for Skilled Personnel	A lack of qualified professionals who can manage and optimize the intersection of technology and tourism management presents a significant barrier.	Elizondo Saltos, D., et al. (2021)

Source: Own Elaboration

5.5. Integration of AI, IoT, and Big Data within SuperHyperStructures for Smart Tourism Management in Punta Cana.

The integration of AI, IoT, and Big Data is revolutionizing smart tourism, and Punta Cana stands to benefit significantly from these advancements. These technologies create a seamless and personalized experience for travelers while enhancing operational efficiency in the region’s tourism industry.

AI algorithms are essential for analyzing vast amounts of data to tailor services based on individual preferences. In Punta Cana, visitors can receive personalized recommendations for activities, dining, and entertainment, making their stay even more memorable. This use of AI helps local businesses better understand customer needs, increasing satisfaction and repeat visits (Bansal, 2024).

IoT devices play a crucial role in connecting various aspects of the tourist experience in Punta Cana. Smart sensors installed in hotels and other facilities monitor occupancy rates, adjust energy consumption, and improve security. For example, IoT technology can automatically regulate lighting and temperature in rooms to optimize energy use, which is essential in a popular destination with high foot traffic. This interconnectedness fosters smarter resource management, contributing to the sustainability goals of the region (Göktaş Kulualp & Sari, 2020).

Big Data empowers destination managers by providing real-time insights into visitor behaviors and trends. In Punta Cana, analyzing peak times and popular attractions allows authorities and business owners to optimize infrastructure, reduce congestion at key sites like Bávaro Beach, and ensure a smoother flow of tourists. These data-driven decisions also help balance tourism growth with environmental sustainability (Ribes & Baidal, 2018).

Together, AI, IoT, and Big Data foster sustainable tourism practices in Punta Cana. They minimize waste while maximizing visitor satisfaction, ensuring that Punta Cana remains a vibrant and eco-friendly destination for years to come (Elizondo Saltos et al., 2021).

Table 4. Impact of AI, IoT, and Big Data in Smart Tourism Management in Punta Cana

Technology	Impact in Punta Cana Tourism	Reference
AI Algorithms	AI analyzes large datasets to tailor services based on individual preferences, offering personalized recommendations to visitors for activities, dining, and entertainment, enhancing their stay.	Bansal, P. (2024)
IoT Devices	IoT devices connect various aspects of the tourist experience, such as hotel occupancy, energy consumption, and security, optimizing resource management and enhancing sustainability.	Göktaş Kulualp, S., & Sari, T. (2020)
Big Data	Big Data provides insights into visitor behavior, helping destination managers optimize infrastructure, reduce congestion, and balance tourism growth with sustainability.	Ribes, J., & Baidal, J. (2018)
Sustainability	The combination of AI, IoT, and Big Data fosters sustainable practices, minimizing waste and maximizing visitor satisfaction, ensuring Punta Cana's continued vibrancy and eco-friendliness.	Elizondo Saltos, D., et al. (2021)

Source: Own Elaboration

6. Future Research Directions

1. **Development of Adaptive Algorithms**: Research adaptive algorithms that respond to real-time data to optimize resource allocation (water, energy, waste management) and enhance the visitor experience within SuperHyperStructures.
2. **User Behavior Simulations:** Study tourist flow dynamics through immersive simulations to improve crowd control strategies and sustainability at key sites like Bávaro Beach, leveraging SuperHyperStructures for better management.
3. **Interdisciplinary Collaboration:** Foster collaboration among technologists, urban planners, and environmentalists to design frameworks within SuperHyperStructures that balance tourism growth with ecological preservation.
4. **Socioeconomic Impacts:** Investigate the socio-economic effects of SuperHyperStructures on local communities in Punta Cana, ensuring that the economic benefits of smart tourism are equitably distributed.
5. **Synergies with Emerging Technologies:** Explore how emerging technologies like blockchain and augmented reality can be integrated into SuperHyperStructures to improve visitor engagement and promote sustainable practices in tourism development.

6. Conclusions

SuperHyperStructures represent a revolutionary approach to managing smart and sustainable tourist destinations. By integrating advanced mathematical models such as SuperHyperGraphs and SuperHyperTopologies, these

systems enable optimized resource allocation, improved decision-making, and enhanced sustainability outcomes. Their ability to handle complexity, uncertainty, and interconnectivity makes them ideal for addressing the challenges of modern tourism ecosystems.

The application of SuperHyperStructures fosters holistic governance by harmonizing real-time data from visitors, businesses, transportation systems, and environmental sensors. This integration enhances operational efficiency while reducing ecological footprints. For example, dynamic data flows can redistribute tourist traffic to alleviate overcrowding and promote lesser-known attractions.
Advanced technologies within SuperHyperStructures allow destinations to tailor visitor experiences using behavioral analytics and AI-powered recommendation engines. This personalization aligns with sustainability goals by encouraging responsible and meaningful travel behaviors.

SuperHyperStructures provide a sophisticated way to represent hierarchical relationships and multi-level interactions within a destination like Punta Cana, capturing its dynamic tourism infrastructure, service providers, environmental systems, and community involvement.

Through Neutrosophic SuperHyperStructures, it is possible to address the uncertain, incomplete, and sometimes contradictory data that often arises in the governance of destinations such as Punta Cana, especially when dealing with seasonal fluctuations, climate resilience, and infrastructure planning.

By applying these advanced mathematical tools, Punta Cana can strengthen its position as a model for smart tourism in the Caribbean. SuperHyperStructures contribute to building resilient strategies, promoting innovation, and ensuring long-term sustainability aligned with global development goals.

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Chapter 25

Neutrosophic Milli Topological Spaces

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Abstract: The main objective of this study is to introduce a new hybrid structure called neutrosophic milli topology. Fuzzy milli topology and intuitionistic milli topology can also be deduced from the neutrosophic milli topology. Based on the neutrosophic milli approximations, we have classified neutrosophic milli topology. Some properties such as neutrosophic milli interior and neutrosophic milli closure are derived.

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1. Introduction

The notion of milli topology was introduced by Ittanagi et al[5]., which was defined in terms of approximations, boundary region, approximations of edges and approximations of boundary regions of a subset of a universe using an equivalence relation on it. The milli topological space consists of maximum of nine elements. Nano topology explored by Thivagar et.al[9, 10] can be described as a collection of nano approximations, a non-empty finite universe and empty set for which equivalence classes are buliding blocks. Also, introduced a new hybrid structure called neutrosophic nano topology, fuzzy nano topology and intuitionistic nano topology. Some properties like neutrosophic nano interior and neutrosophic nano closure are derived. One of the interesting generalizations of the theories[1-4, 6-8] of fuzzy sets and intuitionistic fuzzy sets is the theory of neutrosophic sets introduced by F. Smarandache. Neutrosophic set is described by three functions: a membership function, indeterminacy function and a nonmembership function that are independently related. The theories of neutrosophic set have achieved greater success in various areas such as medical diagnosis, database, topology, image processing and decision making problem.

While the neutrosophic set is a powerful tool to deal with indeterminate and inconsistent data, the theory of rough set is a powerful mathematical tool to deal with incompleteness. Neutrosophic sets and rough sets are two different topics, none conflicts the other. The main objective of this study is to introduce a new hybrid intelligent structure called neutrosophic milli topology. The significance of introducing hybrid structures is that the computational techniques, based on any one of these structures alone, will not always yield the best results but a fusion of two or more of them can often give better results.

While milli topology provides a framework for studying approximations, edges, and boundary regions of subsets in an approximation space, it is inherently limited when dealing with **indeterminate or inconsistent information**. Classical milli topology assumes crisp boundaries once equivalence classes are fixed, which is not always realistic in practical scenarios.

Fuzzy milli topology improves this by incorporating membership degrees, but it cannot capture indeterminacy explicitly it only distinguishes between degrees of belonging and non-belonging.

Intuitionistic milli topology extends this by adding non-membership functions, yet it still lacks the ability to handle **independent indeterminacy**, i.e., situations where information about an element is incomplete, contradictory, or unknown.

In real-world applications such as **geographical information systems (GIS), medical diagnosis, and image processing**, one often encounters data where the state of an element cannot be described solely by membership and non-membership. For example, a medical test result may be partially positive, partially negative, and partially indeterminate due to measurement errors or missing data.

Thus, there is a genuine need to extend milli topology into the **neutrosophic domain**, where three independent parameters membership (truth), non-membership (falsity), and indeterminacy allow a richer and more flexible modeling of uncertainty. The proposed **neutrosophic milli topology** addresses this gap by generalizing milli topology and providing a more robust framework for handling **indeterminate, incomplete, and inconsistent information**.

The rest of this chapter is organized as follows. Some preliminary concepts required in our work are briefly recalled in section 2. In section 3, the concept of neutrosophic milli topology is investigated. Section 4 concludes the chapter with some properties on neutrosophic milli interior and neutrosophic milli closure and the conclusion is given in Section 5.

2. Preliminaries

The following recalls requisite ideas and preliminaries necessitated in the sequel of our work.

Definition 2.1[9]: Let U be a non-empty set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. The pair (U, R) is called the approximation space. Let X be a subset of U.

i). The lower approximation of X with respect to R is the set of all objects,

which can be for certain classified as X with respect to R and it is denoted by $\underline{R}(X)$ and is defined as $\underline{R}(X) = \{x \in U : [x]_R \subseteq X\}$, where $[x]_R$ denotes the equivalence class determined by x.

ii). The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $\overline{R}(X)$ and is defined as $\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \phi\}$.

Definition 2.2[9]: Let (U, R) be an approximation space and R be an equivalence relation on U. Let X be a subset of U.

i). The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not-X with respect to R and it is denoted by $BN(X)$ or $B(X)$. It is defined as $B(X) = \overline{R}(X) - \underline{R}(X)$.

ii). The set X is said to be rough with respect to R if $\overline{R}(X) \neq \underline{R}(X)$. That is, if $B(X) \neq \phi$.

Definition 2.3[9]: Let (U, R) be an approximation space and R be an equivalence relation on U. Let X be a subset of U.

i). The internal edge (lower edge) of X with respect to R and it is denoted by $\underline{E}(X)$. It is defined as set $\underline{E}(X) = X - \underline{R}(X)$

ii). The external edge (uper edge) of X with respect to R and it is denoted by $\overline{E}(X)$. It is defined as set $\overline{E}(X) = \overline{R}(X) - X$.

Definition 2.4[5]: Let U be a non-empty set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. The pair (U, R) is called the approximation space. Let X be a subset of U.

i). The lower boundary region of X with respect to R is denoted by $\underline{B}(X)$ and is defined as $\underline{B}(X) = \overline{R}(X) - \underline{E}(X)$.

ii). The upper boundary region of X with respect to R is denoted by $\overline{B}(X)$ and is defined as $\overline{B}(X) = \overline{R}(X) - \overline{E}(X)$.

Defition 2.5[5]: Let U be an universe of objects and R be an equivalence relation on U and $\tau_M(X) = \{U, \phi, \underline{R}(X), \overline{R}(X), B(X), \underline{E}(X), \overline{E}(X), \underline{B}(X), \overline{B}(X\}$. Where $X \subseteq U$, $\tau_M(X)$ satifies the following axioms:

i) U and $\phi \in \tau_M(X)$

ii) The union of the elements of any sub-collection of $\tau_M(X)$ is in $\tau_M(X)$.

iii) The intersection of the elements of any finite sub-collection of $\tau_M(X)$ is in $\tau_M(X)$.

That is, $\tau_M(X)$ forms a topology on U called the milli topology on U with respect to X. We call $(U, \tau_M(X))$ as the milli topological space. The elements of $\tau_M(X)$ are called milli-open sets, denoted as $MO(X)$.

Theorem 2.6[5]: Let U be a non-empty universe and $X \subseteq U$. Let $\tau_M(X)$ be the milli topology on U with respect to X, then $[\tau_M(X)]^C$ whose elements are A^C, $A \in \tau_M(X)$ is a milli topology on U.

Defition 2 .7[5]: Let $(U, \tau_R(X))$ is a milli topological space with respect to X where $X \subseteq U$ and let $A \subseteq U$, (i) The milli interior of A is defined as the union of all milli open subsets contained in A and it is denoted by $Mint(A)$. That is $Mint(A)$ is the largest milli open subset contained in A.

(ii) the milli closure of A is defined as the intersection of all milli closed sets containing A and it is denoted by $Mcl(A)$. That is $Mcl(A)$ is the smallest milli closed set containing A.

Definition 2.8[10]: Let U be a non-empty, universe of objects and R be an equivalence relation on U. Let $X \subseteq U$. Let $\tau_R(X) = \{U, \phi, \underline{R}(X), \overline{R}(X), B(X)\}$. Then $\tau_R(X)$ is a topology on U called as the nano topology with respect to X.
Definition 2.9[11]: Let U be a non-empty finite universe, R be an equivalence relation on U and F be a neutrosophic set in U and if the collection $\tau_N(F) = \{0_N, 1_N, \underline{N}(F), \overline{N}(F), BN(F), \underline{E}(F)\}$ forms a topology the it is said to be a neutrosophic nano topology. We call $(U, \tau_N(F))$, as the neutrosophic nano topological space. The elements of $\tau_N(F)$ are called neutrosophic nano open sets.
Definition 2.10[3]: Let X be a non-empty set. A fuzzy set A is an object having the form $A = \{< x : \mu_A(x), x \in X\}$, where $0 \leq \mu_A(x) \leq 1$ represent the degree of membership of each $x \in X$ to the set A.
Definition 2.11[2]: Let X be a non-empty set. An intuitionstic set A is of the form $A = \{< x : \mu_A(x), \gamma_A(x), x \in X\}$, where $\mu_A(x)$ and $\gamma_A(x)$ represent the degree of membership function and the degree of non membership respectively of each $x \in X$ to the set A and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$.
Definition 2.12[7]: Let X be an universe of discourse with a generic element in X denoted by x, the neutrosophic set is an object having the form $A = \{< x : \mu_A(x), \sigma_A(x), \gamma_A(x) >: x \in X\}$, where the functions $\mu, \sigma, \gamma : X \Rightarrow [0, 1]$ define respectively the degree of membership or truth, the degree of indeterminancy, and the degree of non-membership (or Falsehood) of the element $x \in X$ to the set A with the condition $-0 \leq \mu_A(x) + \sigma_A(x) + \gamma_A(x) \leq 3$.

3. Neutrosophic Milli Topological Spaces

In this section we introduce the notion of neutrosophic milli topology by means of neutrosophic milli lower approximation, neutrosophic milli upper approximation, neutrosophic milli boundary region, neutrosophic milli lower edge, neutrosophic milli upper edge, neutrosophic milli lower boundary region and neutrosophic milli upper boundary region. From Neutrosophic milli topology we have also defined and deduced intuitionistic milli topology and fuzzy milli topology. Neutrosophic milli topological space is denoted by $(U, \tau_{NM}(F))$.
Definition 3.1: Let U be a non-empty finite universe and R be an equivalence relation on U. Let $N(F)$ be a neutrosophic set in U with the membership function μ_F, the indeterminacy function σ_F and the non-membership function ν_F. The neutrosophic milli lower approximation, neutrosophic milli upper approximation, neutrosophic milli boundary, neutrosophic milli lower edge, neutrosophic milli upper edge, neutrosophic milli lower boundary region and neutrosophic milli upper boundary region of $N(F)$ in the approximation space (U, R) denoted by $\underline{N}(F)$, $\overline{N}(F)$, $B(F)$, $\underline{E}(F)$, $\overline{E}(F)$, $\underline{B}(F)$ and $\overline{B}(F)$ are respectively defined as follows:
(i) The neutrosophic milli lower approximation of $N(F)$ with respect to equivalence class R is denoted by $\underline{N}(F)$ and defined as follows:

$\underline{N}(F) = \{< x,\ \mu_{\underline{R}(F)}(x),\ \sigma_{\underline{R}(F)}(x),\ \nu_{\underline{R}(F)}(x) >: y \in [x]_R, x \in U\}$.
where $\mu_{\underline{R}(F)}(x) = \wedge_{y \in [x]R}\ \{\mu_F(y)\}$, $\sigma_{\underline{R}(F)}(x) = \wedge_{y \in [x]R}\ \{\sigma_F(y)\}$,
$\nu_{\underline{R}(F)}(x) = \vee_{y \in [x]R}\ \{\nu_F(y)\}$.

(ii) The neutrosophic milli upper approximation of $N(F)$ with respect to equivalence class R is denoted by $\overline{N}(F)$ and defined as follows:

$\overline{N}(F) = \{< x,\ \mu_{\overline{R}(F)}(x),\ \ \sigma_{\overline{R}(F)}(x),\ \ \nu_{\overline{R}(F)}(x) >: y \in [x]_R, x \in U\}$.

where $\mu_{\overline{R}(F)}(x) = \vee_{y\in[x]R}\ \{\mu_F(y)\}$, $\sigma_{\overline{R}(F)}(x) = \vee_{y\in[x]R}\ \{\sigma_F(y)\}$, $\nu_{\overline{R}(F)}(x) = \wedge_{y\in[x]R}\ \{\nu_F(y)\}$.

(iii) The neutrosophic milli boundary region of $N(F)$ with respect to the equivalence class R is denoted by $B(F)$ and defined as $B(F) = \overline{N}(F) - \underline{N}(F)$.

(iv) The neutrosophic milli lower edge of $N(F)$ with respect to the equivalence class R is denoted by $\underline{E}(F)$ and defined as $\underline{E}(F) = N(F) - \underline{N}(F)$.

(v) The neutrosophic milli upper edge of $N(F)$ with respect to the equivalence class R is denoted by $\overline{E}(F)$ and defined as $\overline{E}(F) = \overline{N}(F) - N(F)$.

(vi) The neutrosophic milli lower boundary region of $N(F)$ with respect to the equivalence class R is denoted by $\underline{B}(F)$ and defined as $\underline{B}(F) = \overline{N}(F) - \underline{E}(F)$.

(vii) The neutrosophic milli upper boundary region of $N(F)$ with respect to the equivalence class R is denoted by $\overline{B}(F)$ and defined as $\overline{B}(F) = \overline{N}(F) - \overline{E}(F)$.

Remark 3.2: Since our main purpose is to construct tools for developing neutrosophic milli topological spaces, we must introduce 0_N, 1_N and certain neutrosophic set operations in U as follows.

Definition 3.3: Let U be a non-empty finite universe and the neutrosophic sets $N(A)$ and $N(B)$ in the form $N(A) = \{< x, \mu_A(x), \sigma_A(x), \nu_A(x) >: x \in U\}$ and $N(B) = \{< x, \mu_B(x), \sigma_B(x), \nu_B(x) >: x \in U\}$, then the following statements hold:

(i) $0_N = \{< x, 0, 0, 1 >: x \in U\}$, $\quad 1_N = \{< x, 1, 1, 0 >: x \in U\}$.

(ii) $N(A) \subseteq N(B)$ iff $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \leq \sigma_B(x)$, $\nu_A(x) \geq \nu_B(x)$ for all $x \in U$.

(iii) $N(A) = N(B)$ iff $N(A) \subseteq N(B)$ and $N(B) \subseteq N(A)$.

(iv) $[N(A)]^C = \{< x,\ \nu_A(x),\ \ 1 - \sigma_A(x),\ \ \mu_A(x) >: x \in U\}$.

(v) $N(A) \cap N(B)$ iff $\mu_A(x) \wedge \mu_B(x)$, $\sigma_A(x) \wedge \sigma_B(x)$, $\nu_A(x) \vee \nu_B(x)$ for all $x \in U$.

(vi) $N(A) \cup N(B)$ iff $\mu_A(x) \vee \mu_B(x)$, $\sigma_A(x) \vee \sigma_B(x)$, $\nu_A(x) \wedge \nu_B(x)$ for all $x \in U$.

(vii) $N(A) - N(B)$ iff $\mu_A(x) \wedge \nu_B(x)$, $\sigma_A(x) \wedge 1 - \sigma_B(x)$, $\nu_A(x) \vee \mu_B(x)$ for all $x \in U$.

Definition 3.4: Let U be a non-empty finite universe, R be an equivalence relation on U and $N(F)$ be a neutrosophic set in U and if the collection $\tau_{NM}(F) = \{0_N, 1_N, \underline{N}(F), \overline{N}(F), B(F), \underline{E}(F), \overline{E}(F), \underline{B}(F), \overline{B}(F)\}$ satisfies the following axioms:

i) 1_N and 0_N belongs to $\tau_{NM}(F)$.

ii) Union of the elements of any subcollection of $\tau_{NM}(F)$ is in $\tau_{NM}(F)$.

iii) Intersection of the elements of any finite subcollection of $\tau_{NM}(F)$ is in $\tau_{NM}(F)$.

That is, $\tau_{NM}(F)$ forms a topology called the neutrosophic milli topology on U with respect to $N(F)$. We call $(U, \tau_{NM}(F))$ as the neutrosophic milli topological space. The elements of $\tau_{NM}(F)$ are called neutrosophic milli open sets, denoted by $NMO(F)$.

Remark 3.5: From **neutrosophic milli topology** we can deduce and define the **fuzzy milli topology** and **intuitionistic milli topology**. Fuzzy milli

topology is obtained by considering the membership values alone whereas in case of intuitionistic milli topology both membership and non membership values are considered.

Definition 3.6: Let U be a non-empty finite universe and R be an equivalence relation on U. Let $I(F)$ be an intuitionistic set in U with the membership function μ_F and the non-membership function ν_F. The intuitionistic milli lower approximation, intuitionistic milli upper approximation, intuitionistic milli boundary, intuitionistic milli lower edge, intuitionistic milli upper edge, intuitionistic milli lower boundary region and intuitionistic milli upper boundary region of $I(F)$ in the approximation space (U, R) denoted by $\underline{I}(F)$, $\overline{I}(F)$, $B_I(F)$, $\underline{E_I}(F)$, $\overline{E_I}(F)$, $\underline{B_I}(F)$ and $\overline{B_I}(F)$ are respectively defined as follows:
(i) $\underline{I}(F) = \{< x, \mu_{\underline{R}(F)}(x), \nu_{\underline{R}(F)}(x) >: y \in [x]_R, x \in U\}$.
(ii) $\overline{I}(F) = \{< x, \mu_{\overline{R}(F)}(x), \nu_{\overline{R}(F)}(x) >: y \in [x]_R, x \in U\}$.
(iii) $B_I(F) = \overline{I}(F) - \underline{I}(F)$.
(iv) $\underline{E_I}(F) = I(F) - \underline{I}(F)$.
(v) $\overline{E_I}(F) = \overline{I}(F) - I(F)$.
(vi) $\underline{B_I}(F) = \overline{I}(F) - \underline{E_I}(F)$.
(v) $\overline{B_I}(F) = \overline{I}(F) - \overline{E_I}(F)$.
where $\mu_{\underline{R_I}(F)}(x) = \wedge_{y \in [x]R} \{\mu_F(y)\}$, $\nu_{\underline{R_I}(F)}(x) = \vee_{y \in [x]R} \{\nu_F(y)\}$,
$\mu_{\overline{R_I}(F)}(x) = \vee_{y \in [x]R} \{\mu_F(y)\}$, $\nu_{\overline{R_I}(F)}(x) = \wedge_{y \in [x]R} \{\nu_F(y)\}$.

Definition 3.7: Let U be a non-empty finite universe, R be an equivalence relation on U and $I(F)$ be an intuitionistic set in U and if the collection $\tau_{IM}(F) = \{0_N, 1_N, \underline{I}(F), \overline{I}(F), B_I(F), \underline{E_I}(F), \overline{E_I}(F), \underline{B_I}(F), \overline{B_I}(F)\}$ forms a topology then it is said to be an intuitionistic milli topology. We call $(U, \tau_{IM}(F))$ as the intuitionistic milli topological space. The elements of $\tau_{IM}(F)$ are called intuitionistic milli open sets.

Definition 3.8: Let U be a non-empty finite universe and R be an equivalence relation on U. Let F be a fuzzy set in U with the membership function μ_F, then the fuzzy milli lower approximation, fuzzy milli upper approximation, fuzzy milli boundary, fuzzy milli lower edge, fuzzy milli upper edge, fuzzy milli lower boundary region and fuzzy milli upper boundary region of F in the approximation space (U, R) denoted by $\underline{F}(F)$, $\overline{F}(F)$, $B_F(F)$, $\underline{E_F}(F)$, $\overline{E_F}(F)$, $\underline{B_F}(F)$ and $\overline{B_F}(F)$ are respectively defined as follows:
(i) $\underline{F}(F) = \{< x, \mu_{\underline{R}(F)}(x) >: y \in [x]_R, x \in U\}$.
(ii) $\overline{F}(F) = \{< x, \mu_{\overline{R}(F)}(x) >: y \in [x]_R, x \in U\}$.
(iii) $B_F(F) = \overline{F}(F) - \underline{F}(F)$.
(iv) $\underline{E_F}(F) = F - \underline{F}(F)$.
(v) $\overline{E_F}(F) = \overline{F}(F) - F$.
(vi) $\underline{B_F}(F) = \overline{F}(F) - \underline{E_F}(F)$.
(v) $\overline{B_F}(F) = \overline{F}(F) - \overline{E_F}(F)$.
where $\mu_{\underline{R_F}(F)}(x) = \wedge_{y \in [x]R} \{\mu_F(y)\}$, $\mu_{\overline{R_F}(F)}(x) = \vee_{y \in [x]R} \{\mu_F(y)\}$.

Definition 3.9: Let U be a non-empty finite universe, R be an equivalence relation on U and F be a fuzzy set in U and if the collection $\tau_{FM}(F) = \{0_N, 1_N, \underline{F}(F), \overline{F}(F), B_F(F), \underline{E_F}(F), \overline{E_F}(F), \underline{B_F}(F), \overline{B_F}(F)\}$ forms

a topology then it is said to be a fuzzy milli topology. We call $(U, \tau_{FM}(F))$ as the fuzzy milli topological space. The elements of $\tau_{FM}(F)$ are called fuzzy milli open sets.

Remark 3.10: Thus from the above definitions of intuitionistic milli and fuzzy milli topologies we can assure that throughout this chapter all the properties and examples also holds good when it is possible for neutrosophic milli topology.

Remark 3.11: Let $(U, \tau_{NM}(F))$ be a neutrosophic milli topological space with respect to $N(F)$ where $N(F)$ is a neutrosophic subset of U. Elements of $[\tau_{NM}(F)]^C$ are called neutrosophic milli closed sets. Therefore, neutrosophic milli closed sets in U are $[\tau_{NM}(F)]^C = \{0_N, 1_N, [\underline{N}(F)]^C, [\overline{N}(F)]^C, [B(F)]^C, [\underline{E}(F)]^C, [\overline{E}(F)]^C, [\underline{B}(F)]^C, [\overline{B}(F)]^C\}$.

Remark 3.12: $[\tau_{NM}(F)]^C$ is called the dual neutrosophic milli topology of $\tau_{NM}(F)$.

Example 3.13: Let $U = \{p_1, p_2, p_3\}$ be the universe of discourse. Let $U/R = \{\{p_1, p_3\}, \{p_2\}\}$ be an equivalence relation on U and
$N(F) = \{< p_1, (0.5, 0.5, 0.4) >, < p_2, (0.4, 0.6, 0.8) >, < p_3, (0.5, 0.6, 0.4) >\}$ be a neutrosophic set on U. Then
$\underline{N}(F) = \{< p_1, (0.5, 0.5, 0.4) >, < p_2, (0.4, 0.6, 0.8) >, < p_3, (0.5, 0.5, 0.4) >\}$,
$\overline{N}(F) = \{< p_1, (0.5, 0.6, 0.4) >, < p_2, (0.4, 0.6, 0.8) >, < p_3, (0.5, 0.6, 0.4) >\}$,
$B(F) = \{< p_1, (0.4, 0.5, 0.5) >, < p_2, (0.4, 0.6, 0.8) >, < p_3, (0.4, 0.5, 0.5) >\}$,
$\underline{E}(F) = \{< p_1, (0.4, 0.5, 0.5) >, < p_2, (0.4, 0.6, 0.8) >, < p_3, (0.4, 0.5, 0.5) >\}$,
$\overline{E}(F) = \{< p_1, (0.4, 0.5, 0.5) >, < p_2, (0.4, 0.6, 0.8) >, < p_3, (0.4, 0.4, 0.5) >\}$,
$\underline{B}(F) = \{< p_1, (0.5, 0.5, 0.4) >, < p_2, (0.4, 0.6, 0.8) >, < p_3, (0.5, 0.5, 0.4) >\}$,
$\overline{B}(F) = \{< p_1, (0.5, 0.5, 0.4) >, < p_2, (0.4, 0.6, 0.8) >, < p_3, (0.5, 0.6, 0.4) >\}$.
The collection $\tau_{NM}(F) = \{0_N, 1_N, \underline{N}(F), \overline{N}(F), B(F), \underline{E}(F), \overline{E}(F), \underline{B}(F), \overline{B}(F)\}$ is a neutrosophic milli topology on U and $[\tau_{NM}(F)]^C$ is also a neutrosophic milli topology on U.

Also $I(F) = \{< p_1, (0.5, 0.4) >, < p_2, (0.4, 0.8) >, < p_3, (0.5, 0.4) >\}$ be an intuitionistic set on U. Then
$\underline{I}(F) = \{< p_1, (0.5, 0.4) >, < p_2, (0.4, 0.8) >, < p_3, (0.5, 0.4) >\}$,
$\overline{I}(F) = \{< p_1, (0.5, 0.4) >, < p_2, (0.4, 0.8) >, < p_3, (0.5, 0.4) >\}$,
$B_I(F) = \{< p_1, (0.4, 0.5) >, < p_2, (0.4, 0.8) >, < p_3, (0.4, 0.5) >\}$,
$\underline{E_I}(F) = \{< p_1, (0.4, 0.5) >, < p_2, (0.4, 0.8) >, < p_3, (0.4, 0.5) >\}$,
$\overline{E_I}(F) = \{< p_1, (0.4, 0.5) >, < p_2, (0.4, 0.8) >, < p_3, (0.4, 0.5) >\}$,
$\underline{B_I}(F) = \{< p_1, (0.5, 0.4) >, < p_2, (0.4, 0.8) >, < p_3, (0.5, 0.4) >\}$,
$\overline{B_I}(F) = \{< p_1, (0.5, 0.4) >, < p_2, (0.4, 0.8) >, < p_3, (0.5, 0.4) >\}$.

Thus $\tau_{IM}(F) = \{0_I, 1_I, \underline{I}(F), \overline{I}(F), B_I(F), \underline{E_I}(F), \overline{E_I}(F), \underline{B_I}(F), \overline{B_I}(F)\}$ is the intuitionistic milli topology.

And also $F = \{< p_1, 0.5 >, < p_2, 0.4 >, < p_3, 0.5 >\}$ be a fuzzy set on U. Then
$\underline{F}(F) = \{< p_1, 0.5 >, < p_2, 0.4 >, < p_3, 0.5 >\}$,
$\overline{F}(F) = \{< p_1, 0.5 >, < p_2, 0.4 >, < p_3, 0.5 >\}$,
$B_F(F) = \{< p_1, 0.4 >, < p_2, 0.4 >, < p_3, 0.4 >\}$,
$\underline{E_F}(F) = \{< p_1, 0.4 >, < p_2, 0.4 >, < p_3, 0.4 >\}$,
$\overline{E_F}(F) = \{< p_1, 0.4 >, < p_2, 0.4 >, < p_3, 0.4 >\}$,

$\underline{B_F}(F) = \{< p_1, 0.5 >, < p_2, 0.4 >, < p_3, 0.5 >\}$,
$\overline{B_F}(F) = \{< p_1, 0.5 >, < p_2, 0.4 >, < p_3, 0.5 >\}$.

Thus $\tau_{FM}(F) = \{0, 1, \underline{F}(F), \overline{F}(F), B_F(F), \underline{E_F}(F), \overline{E_F}(F), \underline{B_F}(F), \overline{B_F}(F)\}$ is the fuzzy milli topology.

Theorem 3.14 Every neutrosophic nano topology is neutrosophic milli topology but the converse need not be true.

Proof: Let U be a non-empty finite universe and R be an equivalence relation on U. Let $N(F)$ be a neutrosophic set in U with the neutrosophic nano topology $\tau_N(F) = \{0_N, 1_N, \underline{N}(F), \overline{N}(F), B(F)\}$. Neutrosophic nano topology is based on the concept of neutrosophic nano lower approximation, neutrosophic nano upper approximation and neutrosophic nano boundary. The neutrosophic nano topology has a maximum of five neutrosophic nano open sets and a minimum of three neutrosophic nano open sets including 0_N and 1_N.

The neutrosophic milli topology $\tau_{NM}(F) = \{0_N, 1_N, \underline{N}(F), \overline{N}(F), B(F), \underline{E}(F), \overline{E}(F), \underline{B}(F), \overline{B}(F)\}$ is based on the concept of neutrosophic milli lower approximation, neutrosophic milli upper approximation, neutrosophic milli boundary and also adding some more neutrosophic milli open sets are neutrosophic milli lower edge, neutrosophic milli upper edge, neutrosophic milli lower boundary region and neutrosophic milli upper boundary region. The neutrosophic milli topology has a maximum of nine neutrosophic milli open sets and a minimum of seven neutrosophic milli open sets including 0_N and 1_N.

Note that the neutrosophic nano open sets 0_N, 1_N, neutrosophic nano lower approximation $\underline{N}(F)$, neutrosophic nano milli upper approximation $\overline{N}(F)$ and neutrosophic nano boundary $B(F)$ are always belong to neutrosophic milli open sets. Therefore, every neutrosophic nano open set is neutrosophic milli open set. Hence every neutrosophic nano topology is a neutrosophic milli topology. That is every neutrosophic nano topological space is a neutrosophic milli topological space.

Example 3.15: In Example 3.13, neutrosophic milli topology is
$\tau_{NM}(F) = \{0_N, 1_N, \underline{N}(F), \overline{N}(F), B(F), \underline{E}(F), \overline{E}(F), \underline{B}(F), \overline{B}(F)\}$.
The neutrosophic nano topology is $\tau_N(F) = \{0_N, 1_N, \underline{N}(F), \overline{N}(F), B(F)\}$.
Nutrosophic milli open sets are: $\{0_N, 1_N, \underline{N}(F), \overline{N}(F), B(F), \underline{E}(F), \overline{E}(F), \underline{B}(F), \overline{B}(F)\}$. Neutrosophic nano open sets are: $\{0_N, 1_N, \underline{N}(F), \overline{N}(F), B(F)\}$.
The set $\overline{B}(F)$ is nutrosophic milli open set but not nutrosophic nano open set.

Definition 3.16: Types of neutrosophic milli topology: Let U be a non-empty universe and $N(F)$ be a neutrosophic set on U. Then the following statements hold:

Type - 1: If $\underline{N}(F) = \overline{N}(F) = 1_N$, then $\tau_{NM}(X) = \{0_N, 1_N\}$, is the indiscrete neutrosophic milli topology on U.

Type - 2: If $\underline{N}(F) = \overline{N}(F) = N(F)$ or $B(F) = 0_N$, then $\tau_{NM}(F) = \{0_N, 1_N, N(F)\}$.

Type - 3: If $\underline{N}(F) = 0_N$ and $\overline{N}(F) = 1_N$, then $\tau_{NM}(X) = \{0_N, 1_N, \underline{E}(F), \overline{E}(F)\}$.

Type - 4: If $\underline{N}(F) = 0_N$ and $\overline{N}(F) \neq 1_N$, then $\tau_{NM}(F) = \{0_N, 1_N, \overline{N}(F), \underline{E}(F), \overline{E}(F)\}$.

Type - 5: If $\underline{N}(F) \neq 0_N$ and $\overline{N}(F) = 1_N$, then

$\tau_{NM}(F) = \{0_N, 1_N, \underline{N}(F), B(F), \underline{E}(F), \overline{E}(F), \underline{B}(F), \overline{B}(F)\}$.

Type - 6: If $\underline{N}(F) \neq \overline{N}(F)$, where $\underline{N}(F) \neq 0_N$ and $\overline{N}(F) \neq 1_N$, then $\tau_{NM}(F) = \{0_N, 1_N, \underline{N}(F), \overline{N}(F), B(F), \underline{E}(F), \overline{E}(F), \underline{B}(F), \overline{B}(F)\}$ is the discrete neutrosophic milli topological space on U.

Proposition 3.17: Let U be a non-empty universe and $N(F)$ be a neutrosophic set on U. If $\tau_{NM}(F)$ is a neutrosophic milli topology on U, then the set $\beta_{NM} = \{1_N, \underline{N}(F), \underline{E}(F), \overline{E}(F)\}$ is a basis for $\tau_{NM}(F)$.

Proof: i) $\bigcup_{N(F)\in\beta_{NM}}(N(F)) = 1_N$.

ii) Consider 1_N and $\underline{N}(F)$ from β_{NM}. Let $N(W) = \underline{N}(F)$, since $1_N \cap \underline{N}(F)$ $= \underline{N}(F)$, $N(W) \subset \{1_N \cap \underline{N}(F)\}$ and every p in $\{1_N \cap \underline{N}(F)\}$ belongs to $N(W)$.

iii) Consider 1_N and $\underline{E}(F)$ from β_{NM}. Let $N(W) = \underline{E}(F)$, since $1_N \cap \underline{E}(F)$ $= \underline{E}(F)$, $N(W) \subset \{1_N \cap \underline{E}(F)\}$ and every p in $\{1_N \cap \underline{E}(F)\}$ belongs to $N(W)$.

iv) Consider 1_N and $\overline{E}(F)$ from β_{NM}. Let $N(W) = \overline{E}(F)$, since $1_N \cap \overline{E}(F)$ $= \overline{E}(F)$, $N(W) \subset \{1_N \cap \overline{E}(F)\}$ and every p in $\{1_N \cap \overline{E}(F)\}$ belongs to $N(W)$.

v) Consider $\underline{N}(X)$ and $\underline{E}(F)$ from β_{NM}, $\underline{N}(F) \cap \underline{E}(F) = 0_N$.

vi) Consider $\underline{N}(F)$ and $\overline{E}(F)$ from β_{NM}, $\underline{N}(F) \cap \overline{E}(F) = 0_N$.

viii) Consider $\underline{E}(F)$ and $\overline{E}(F)$ from β_{NM}, $\underline{E}(F) \cap \overline{E}(F) = 0_N$.

Thus β_{NM} is a basis for $\tau_{NM}(F)$.

Example 3.18: In Example 3.13, neutrosophic milli topology is $\tau_{NM}(F) = \{0_N, 1_N, \underline{N}(F), \overline{N}(F), B(F), \underline{E}(F), \overline{E}(F), \underline{B}(F), \overline{B}(F)\}$. Then the set $\beta_{NM} = \{1_N, \underline{N}(F), \underline{E}(F), \overline{E}(F)\}$ is a basis for $\tau_{NM}(F)$.

4. Neutrosophic Milli Closure and Interior

In this section we have defined neutrosophic milli closure and neutrosophic milli interior on neutrosophic milli topological space. Based on this we also prove some properties.

Definition 4.1: Let $(U, \tau_{NM}(F))$ be a neutrosophic milli topological space and $N(A) = \{< x, \mu_A(x), \sigma_A(x), \nu_A(x) >: x \in U\}$ be a neutrosophic set in U.

i) Neutrosophic milli interior of $N(A)$ is denoted by $N_M int(N(A))$ and defined as $N_M int(N(A)) = \bigcup\{N(G) : N(G) \text{ is } NMO(F) \text{ and } N(G) \subseteq N(A)\}$.

That is $N_M int(N(A))$ is the union of all neutrosophic milli open subsets of $N(A)$. Clearly $N_M int(N(A))$ is the largest neutrosophic milli open subset of $N(A)$.

ii) Neutrosophic milli closure of $N(A)$ is denoted by $N_M cl(N(A))$ and is defined as $N_M cl(N(A)) = \bigcap\{N(H) : N(H) \text{ is } NMC(F) \text{ and } N(A) \subseteq N(H)\}$.

That is $N_M cl(N(A))$ is the intersection of all neutrosophic milli closed sets containing $N(A)$. Clearly $N_M cl(N(A))$ is the smallest neutrosophic milli closed set containing $N(A)$.

Theorem 4.2[5]: Let $(U, \tau_M(X))$ be a milli topological space with respect to $X \subseteq U$ then $Mcl(x) = U$.

Remark 4.3: The above Theorem 4.2, need not be true for all neutrosophic milli topological space $\tau_{NM}(F)$ with respect to $N(F)$ where $N(F)$ is a neutrosophic subset of U. That is $N_M cl(N(A))$ need not be equal to U which can be shown by the following example.

Example 4.4: Let $U = \{p_1, p_2, p_3, p_4, p_5\}$ be the universe of discourse. Let $U/R = \{\{p_1, p_4\}, \{p_2, p_3\}, \{p_5\}\}$ be an equivalence relation on U and $N(F) = \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.4, 0.6, 0.2) >\}$ be a neutrosophic set on U. Then
$\underline{N}(F) = \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.4, 0.6, 0.2) >\}$,
$\overline{N}(F) = \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.4, 0.6, 0.2) >\}$,
$B(F) = \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.2, 0.4, 0.4) >\}$,
$\underline{E}(F) = \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.2, 0.4, 0.4) >\}$,
$\overline{E}(F) = \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.2, 0.4, 0.4) >\}$,
$\underline{B}(F) = \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.4, 0.6, 0.2) >\}$,
$\overline{B}(F) = \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.4, 0.6, 0.2) >\}$.
The collection $\tau_{NM}(F) = \{0_N, 1_N, \underline{N}(F), \overline{N}(F), B(F), \underline{E}(F), \overline{E}(F), \underline{B}(F), \overline{B}(F)\}$ is a neutrosophic milli topology on U. We observe that
$N(F) = \underline{N}(F) = \overline{N}(F) = \underline{B}(F) = \overline{B}(F)$ and $B(F) = \underline{E}(F) = \overline{E}(F)$.
Therefore $\tau_{NM}(F) = \{0_N, 1_N, N(F), B(F)\}$. Now we have
$\tau_{NM}(F) = \{0_N, 1_N, \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.4, 0.6, 0.2) >\}, \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.2, 0.4, 0.4) >\}$ which is a neutrosophic milli topology on U. Neutrosophic milli closed sets are:
$[\tau_{NM}(F)]^C = \{1_N, 0_N, \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.4, 0.6, 0.2) >\}, \{< p_1, (0.2, 0.3, 0.4) >, < p_4, (0.2, 0.3, 0.4) >, < p_5, (0.2, 0.4, 0.4) >\}$.
Here $N_M cl(N(F)) \neq U$.

Theorem 4.5: Let $(U, \tau_{NM}(F))$ be a neutrosophic milli topological space with respect to $N(F)$ where $N(F)$ is a neutrosophic subset of U. Let $N(A)$ and $N(B)$ be neutrosophic subsets of U. Then the following statements hold:
(i) $N(A) \subseteq N_M cl(N(A))$.
(ii) $N(A)$ is neutrosophic milli closed if and only if $N_M cl(N(A)) = N(A)$.
(iii) $N_M cl(0_N) = 0_N$ and $N_M cl(1_N) = 1_N$.
(iv) $N(A) \subseteq N(B) \Rightarrow N_M cl(N(A)) \subseteq N_M cl(N(B))$.
(v) $N_M cl\{N(A) \cup N(B)\} = N_M cl(N(A)) \cup N_M cl(N(B))$.
(vi) $N_M cl\{N(A) \cap N(B)\} \subseteq N_M cl(N(A)) \cap N_M cl(N(B))$.
(vii) $N_M cl(N_M cl(N(A))) = N_M cl(N(A))$.

Proof : (i) By the definition of neutrosophic milli closure, $N(A) \subseteq N_M cl(N(A))$.
(ii) If $N(A)$ is neutrosophic milli closed, then $N(A)$ is the smallest neutrosophic milli closed set containing itself and hence $N_M cl(N(A)) = N(A)$. Conversely, if $N_M cl(N(A)) = N(A)$, then $N(A)$ is the smallest neutrosophic milli closed set containing itself and hence $N(A)$ is neutrosophic milli closed.
(iii) Since 0_N and 1_N are neutrosophic milli closed in $(U, \tau_{NM}(F))$, $N_M cl(0_N) = 0_N$ and $N_M cl(1_N) = 1_N$.
(iv) If $N(A) \subseteq N(B)$, since $N(B) \subseteq N_M cl(N(B))$, then $N(A) \subseteq N_M cl(N(B))$. That is, $N_M cl(N(B))$ is a neutrosophic milli closed set containing $N(A)$. But $N_M cl(N(A))$ is the smallest neutrosophic milli closed set containing $N(A)$. Therefore, $N_M cl(N(A)) \subseteq N_M cl(N(B))$.
(v) Since $N(A) \subseteq \{N(A) \cup N(B)\}$ and $N(B) \subseteq \{N(A) \cup N(B)\}$, $N_M cl(N(A)) \subseteq N_M cl\{N(A) \cup N(B)\}$ and $N_M cl(N(B)) \subseteq N_M cl\{N(A) \cup N(B)\}$. Therefore, $N_M cl(N(A)) \cup N_M cl(N(B)) \subseteq N_M cl\{N(A) \cup N(B)\}$. By the fact that $\{N(A) \cup N(B)\} \subseteq N_M cl(N(A)) \cup N_M cl(N(B))$, and since $N_M cl\{N(A) \cup N(B)\}$

is the smallest milli closed set containing $\{N(A) \cup N(B)\}$, so $N_M cl\{N(A) \cup N(B)\} \subseteq N_M cl(N(A) \cup N_M cl(N(B))$. Thus, $N_M cl\{N(A) \cup N(B)\} = N_M cl(N(A)) \cup N_M cl(N(B))$.
(vi) Since $\{N(A) \cap N(B)\} \subseteq N(A)$ and $\{N(A) \cap N(B)\} \subseteq N(B)$, $N_M cl\{N(A) \cap N(B))\} \subseteq N_M cl(N(A)) \cap N_M cl(N(B))$.
(vii) Since $N_M cl(N(A))$ is neutrosophic milli closed, then we have $N_M cl(N_M cl(N(A))) = N_M cl(N(A))$.

Theorem 4.6: Let $(U, \tau_{NM}(F))$ be a neutrosophic milli topological space with respect to $N(F)$ where $N(F)$ is a neutrosophic subset of U. Let $N(A)$ be a neutrosophic subset of U. Then
(i) $1_N - N_M int(N(A)) = N_M cl(1_N - N(A))$.
(ii) $1_N - N_M cl(N(A)) = N_M int(1_N - N(A))$.

Proof: i) Let $p \in [1_N - N_M int(N(A))]$. This means that $p \notin N_M int(N(A))$. Therefore, $N(G) \not\subseteq N(A)$, for every neutrosophic milli open set $N(G)$ containing p. That is, $N(G) \cap \{1_N - N(A)\} \neq 0_N$ for every neutrosophic milli open set $N(G)$ containing p. Thus $p \in N_M cl\{1_N - N(A)\}$. It follows that $\{1_N - N_M int(N(A))\} \subseteq N_M cl\{1_N - N(A)\}$. Conversely, if $p \in N_M cl\{1_N - N(A)\}$, then $N(G) \cap \{1_N - N(A)\} \neq 0_N$ for every neutrosophic milli open set $N(G)$ containing p. That is, $N(G) \not\subseteq N(A)$ for every neutrosophic milli open set $N(G)$ containing p such that $p \notin N_M int(N(A))$. Hence $p \in \{1_N - N_M int(N(A)\}$. Thus $N_M cl\{1_N - N(A)\} \subseteq \{1_N - N_M int(N(A)\}$. This means that $1_N - N_M int(N(A)) = N_M cl\{1_N - N(A)\}$.
ii) Proof is similar to that of (i).

Remark 4.7: Taking complements on either side of (i) and (ii) Theorem 4.6, we get $N_M int(N(A)) = 1_N - \{N_M cl(1_N - N(A))\}$ and $N_M cl(N(A)) = 1_N - \{N_M int(1_N - N(A))\}$.

Example 4.8: Let $U = \{p_1, p_2, p_3\}$ and $U/R = \{\{p_1, p_2\}, \{p_3\}\}$. Let $N(F) = \{< p_1, (0.4, 0.5, 0.5) >, < p_2, (0.4, 0.5, 0.5) >, < p_3, (0.5, 0.5, 0.5) >\}$ be a neutrosophic set on U.Then
$\underline{N}(F) = \{< p_1, (0.4, 0.5, 0.5) >, < p_2, (0.4, 0.5, 0.5) >, < p_3, (0.5, 0.5, 0.5) >\}$,
$\overline{N}(F) = \{< p_1, (0.4, 0.5, 0.5) >, < p_2, (0.4, 0.5, 0.5) >, < p_3, (0.5, 0.5, 0.5) >\}$,
$B(F) = \{< p_1, (0.4, 0.5, 0.5) >, < p_2, (0.4, 0.5, 0.5) >, < p_3, (0.5, 0.5, 0.5) >\}$,
$\underline{E}(F) = \{< p_1, (0.4, 0.5, 0.5) >, < p_2, (0.4, 0.5, 0.5) >, < p_3, (0.5, 0.5, 0.5) >\}$,
$\overline{E}(F) = \{< p_1, (0.4, 0.5, 0.5) >, < p_2, (0.4, 0.5, 0.5) >, < p_3, (0.5, 0.5, 0.5) >\}$,
$\underline{B}(F) = \{< p_1, (0.4, 0.5, 0.5) >, < p_2, (0.4, 0.5, 0.5) >, < p_3, (0.5, 0.5, 0.5) >\}$,
$\overline{B}(F) = \{< p_1, (0.4, 0.5, 0.5) >, < p_2, (0.4, 0.5, 0.5) >, < p_3, (0.5, 0.5, 0.5) >\}$.
The collection $\tau_{NM}(F) = \{1_N, 0_N, \underline{N}(F), \overline{N}(F), B(F), \underline{E}(F), \overline{E}(F), \underline{B}(F), \overline{B}(F)\}$ is a neutrosophic milli topology on U. We observe that $N(F) = \underline{N}(F) = \overline{N}(F) = \underline{B}(F) = \overline{B}(F) = B(F) = \underline{E}(F) = \overline{E}(F) = N(F)$. Therefore $\tau_{NM}(F) = \{0_N, 1_N, N(F)\}$. Now we have $\tau_{NM}(F) = \{0_N, 1_N, \{< p_1, (0.4, 0.5, 0.5) >, < p_2, (0.4, 0.5, 0.5) >, < p_3, (0.5, 0.5, 0.5) >\}$ is a neutrosophic milli topology on U. Neutrosophic milli closed sets are: $[\tau_{NM}(F)]^C = \{1_N, 0_N, \{< p_1, (0.4, 0.5, 0.5) >, < p_2, (0.4, 0.5, 0.5) >, < p_3, (0.5, 0.5, 0.5) >\}$.

If $N(A) = \{< p_1, (0.7, 0.6, 0.5) >, < p_2, (0.3, 0.4, 0.5) >, < p_3, (0.7, 0.5, 0.5) >\}$, then $[N_M cl(N(A))]^C = 1_N$, $N_M cl(1_N - N(A)) = 1_N$. That is

$\{1_N - N_M int(N(A))\} = N_M cl(1_N - N(A))$. Also
$\{1_N - N_M cl(N(A))\} = N_M int(1_N - N(A)) = 0_N$.
Theorem 4.9: Let $(U, \tau_{NM}(F))$ be a neutrosophic milli topological space with respect to $N(F)$ where $N(F)$ is a neutrosophic subset of U. Let $N(A)$ and $N(B)$ be neutrosophic subsets of U. Then the following statements hold:
(i) $N(A)$ is neutrosophic milli open if and only if $N_M int(N(A)) = N(A)$.
(ii) $N_M int(0_N) = 0_N$ and $N_M int(1_N) = 1_N$.
(iii) $N(A) \subseteq N(B) \Rightarrow N_M int(N(A)) \subseteq N_M int(N(B))$.
(iv) $N_M int(N(A)) \cup N_M int(N(B)) \subseteq N_M int\{N(A) \cup N(B)\}$.
(v) $N_M int\{N(A) \cap N(B)\} = N_M int(N(A)) \cap N_M int(N(B))$.
(vi) $N_M int(N_M int(N(A))) = N_M int(N(A))$.

Proof: (i) $N(A)$ is neutrosophic milli open if and only if $1_N - N(A)$ is neutrosophic milli closed, if and only if $N_M cl\{1_N - N(A)\} = 1_N - N(A)$, if and only if $1_N - N_M int\{1_N - N(A)\} = N(A)$ if and only if $N_M int(N(A)) = N(A)$, by Remark 4.7.
(ii) Since 0_N and 1_N are neutrosophic milli open,
$N_M int(0_N) = 0_N$ and $N_M int(1_N) = 1_N$.
(iii) $N(A) \subseteq N(B) \Rightarrow 1_N - N(B) \subseteq 1_N - N(A)$.
Therefore, $N_M cl\{1_N - N(B)\} \subseteq N_M cl\{1_N - N(A)\}$.
That is, $1_N - \{N_M cl(1_N - N(B)\} \subseteq 1_N - \{N_M cl(1_N - N(A))\}$.
Hence $N_M int(N(A)) \subseteq N_M int(N(B))$.

Proof of (iv), (v) and (vi) follows similarly from Theorem 4.6 and Remark 4.7.

5. Conclusion

This chapter has introduced the concept of **neutrosophic milli topology**, a new hybrid topological framework that extends the existing notions of milli, fuzzy, and intuitionistic milli topologies by incorporating neutrosophic sets. Through detailed definitions, properties, and examples, we demonstrated how neutrosophic milli topological spaces provide a richer mathematical structure for handling uncertainty, vagueness, and indeterminacy. In particular, the proposed framework generalizes earlier approaches by allowing each element to be described not only by membership and non-membership degrees but also by an independent measure of indeterminacy. This additional degree of freedom makes neutrosophic milli topology especially suitable for real-world problems where information is incomplete, inconsistent, or partially unknown. The theoretical development included the construction of neutrosophic milli open and closed sets, the formulation of closure and interior operators, the identification of different types of neutrosophic milli topologies, and proofs of fundamental properties such as monotonicity, duality, and idempotence. Importantly, it was shown that every neutrosophic nano topology is a neutrosophic milli topology, although the converse need not hold, thereby establishing the generality of the new framework.

Despite these contributions, the present study also has certain limitations.

The results are formulated in the context of finite universes and rely heavily on equivalence relations for constructing approximation spaces. This restriction, while suitable for illustrative purposes, limits the scope of the theory when applied to larger or continuous domains. Furthermore, the applications section has been outlined only at a conceptual level, without detailed case studies or computational experiments to demonstrate the practical advantages of neutrosophic milli topology over fuzzy or intuitionistic versions. These limitations, however, open promising avenues for further research. Future work may focus on extending the framework to **infinite or continuous universes**, exploring **category-theoretic generalizations**, and developing efficient algorithms for computing neutrosophic milli approximations in large-scale data environments. Applications in **geographical information systems (GIS), remote sensing, medical diagnosis, image processing**, and **decision-making under uncertainty** appear particularly promising, as these domains naturally involve spatial regions or datasets with indeterminate boundaries and incomplete information.

In conclusion, neutrosophic milli topology represents a significant step forward in the ongoing effort to develop mathematical tools that can rigorously accommodate uncertainty and indeterminacy. By situating itself at the intersection of neutrosophic set theory and milli topology, it provides both a theoretical contribution to topology and a practical modeling tool for real-world systems. With further refinement, generalization, and application, this framework has the potential to become an influential component of the broader field of soft computing and uncertain information modeling.

Conflict of Interest: The authors of this work declare that they have no conflicts of interest.

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This volume presents some original research on SuperHyperStructures and SuperHyperTopologies, new branches of mathematics with applications in applied sciences. These frameworks extend classical notions of sets, graphs and topologies by incorporating uncertainty, indeterminacy and multivalued membership, rooted in neutrosophic and Plithogenic logic. They provide non-rigorous methods for the analysis of complex systems in which ambiguity and dynamic interactions are fundamental.

The n-th powersets that are the bases o the SuperHyperStructure, together with their particular cases such as: SuperHyperAlgebra (endowed with SuperHyperOperations and SuperHyperAxioms), SuperHyperGraph (including SuperHyperTree), SuperHyperSoft Set, SuperHyperFunction, SuperHyperTopology (all built on the powersets of P(H), or Pn(H), for $n \geq 1$) and SuperHyperTopology were all founded by Smarandache and developed between 2016 - 2024.

The book is largely deals with 3 main subject-matters. The first one introduces mathematical foundations, including SuperHyperTopologies, Neutrosophic and Plithogenic extensions, fuzzy and soft set frameworks, and links with Łukasiewicz algebras. The second one develops structural perspectives through SuperHyperGraphs, weighted hyperedges, fuzzy neural networks, and neutrosophic automata. The third subject-matter illustrates interdisciplinary applications in medicine, finance, agriculture, tourism, economics, and data science.

An important feature of the volume is its inter-disciplinary orientation. Mathematical theories are linked to practical problems, showing how SuperHyper concepts can support decision-making, optimize resources and model uncertainty in different fields. By combining fuzzy, neutrosophic and plithogenic logics within hyperstructural frameworks, the book highlights innovative approaches to indeterminate and multilayered phenomena.

The subject-matters and content of this volume is aimed not only at researchers but also graduate students and professionals in mathematics, computer science and related disciplines. It offers a comprehensive reference and several practical research avenues, paving the way for further interdisciplinary research and applications.

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